Characterization of polynomials satisfying the Super Haar Condition

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ABSTRACT

The scaling function for the Super Haar wavelet is a linear combination of shifts in the Haar scaling function; the coefficients of this linear combination are assumed to be integers. If the scaling function satisfies the dilation equation the coefficients are said to be Super Haar Admissible. It has been shown that the z transform of Super Haar Admissible coefficients results in a polynomial that satisfies certain conditions. We define a related condition, which we call the Super Haar Condition and show that cyclotomic polynomials of odd order satisfy it. Further, dilation coefficients associated with such polynomials can immediately be found from relations among the cyclotomic polynomials. Using these results, a large class of Super Haar Admissible coefficients is identified and we conjecture that this class includes all admissible coefficients. We discuss applications to denoising and present an example.

Keywords: Haar wavelet, Super Haar wavelet, scaling function, dilation equation, cyclotomic polynomials, denoising.

1. THE SUPER HAAR CONDITION

The Haar scaling function $\phi_H(t)$ is defined by

$$\phi_H(t) = 1, \quad t \in [0,1) \quad \text{and} \quad \phi_H(t) = 0, \quad t \notin [0,1) \quad (1)$$

The Super Haar scaling function, $\phi(t)$ is a weighted sum of shifts in the Haar scaling function

$$\phi(t) = \sum_{k=0}^n s_k \phi_H(t - k) \quad (2)$$

where the scaling coefficients s_k are assumed to be integers; $\phi(t)$ must satisfy a dilation equation

$$\phi(t) = 2 \sum_{k=0}^m d_k \phi(2t - k) \quad (3)$$

It is shown ¹ that the dilation coefficients d_k , if they exist, can be found from the scaling coefficients s_k . Reference ¹ takes the Fourier Transform of (1), (2) and (3) and defines $S(z) = \sum_{k=0}^n s_k z^k$, $D(z) = \sum_{k=0}^m d_k z^k$, where $z = \exp(-j\theta)$. Thus the polynomial $S(z)$ is identified with the scaling coefficients and $D(z)$ with the dilation coefficients. Using algebraic manipulation of the Fourier Transforms of (1), (2) and (3), ¹ derives

$$D(z) = \frac{S(z^2)(z+1)}{2S(z)} \quad (4)$$

Hence $D(z)$, if it exists, is determined from $S(z)$ by (4). This paper addresses the question of finding a class of polynomials $S(z)$ for which it is guaranteed that a corresponding $D(z)$ does exist. In order for $D(z)$ to exist, the expression on the right side of (4) must be a polynomial, rather than a polynomial plus a remainder term. Also, since $z = \exp(-j\theta)$, all the roots of $S(z)$ must be on the complex unit circle. In addition, we recall that all the scaling coefficients s_k are assumed to be integers. If

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$D(z)$ does exist for integral scaling coefficients s_k we say these coefficients are *Super Haar Admissible* and the polynomial $S(z)$ is called a Super Haar Admissible scaling polynomial.

We wish to define the Super Haar Condition free of extraneous details. To do this we impose some additional restrictions. If $S(z)$ satisfies all the aforementioned criteria, it is obvious that $kS(z)$ will also, where k is any non-zero integer. Thus it is reasonable to assume that $S(z)$ is a “monic” polynomial: that is, the coefficient of the highest power of z is one. That is, in the expression $S(z) = \sum_0^n s_k z^k$, we assume $s_n = 1$. Finally, suppose that

$$R(z) = \frac{S(z^2)}{S(z)} \quad (5)$$

is a polynomial. Then, by (4), $D(z) = R(z)(z+1)/2$ will also be a polynomial. It may be that even if $R(z)$ is not a polynomial that $D(z)$ will still be a polynomial. However, for simplicity, we will assume $R(z)$ is a polynomial. We formalize all the above as follows.

Definition 1: Suppose $S(z)$ is a non-zero polynomial,

$$S(z) = \sum_0^n s_k z^k \quad (6)$$

and let

$$R(z) = \frac{S(z^2)}{S(z)} \quad (7)$$

If

- (i) Each s_k is an integer and $s_n = 1$.
- (ii) The roots z_j of $S(z)$ are all on the complex unit circle.
- (iii) $R(z)$ is a polynomial.

then we say that $S(z)$ satisfies the Super Haar Condition.

Clearly, every polynomial which satisfies the Super Haar Condition has coefficients which are admissible as Super Haar scaling coefficients, but the converse need not be true. Thus the Super Haar Condition is sufficient but not necessary for the admissibility of the associated scaling coefficients.

2. CYCLOTOMIC POLYNOMIALS

For convenience, we define, for positive integer n

$$r_n = \exp(j2\pi / n) \quad (8)$$

and for any integer m

$$r_n^m = (r_n)^m = \exp(j2\pi \frac{m}{n}) \quad (9)$$

The following is standard terminology from number theory. For integers n and m , where n is not zero, if the division of n into m results in an integer, we say n divides m . The largest integer that divides into both n and m is called the Greatest Common Divisor (GCD) of n and m . The following notation is used:

$$\text{GCD of } n \text{ and } m = (n, m) \quad (10)$$

If $(n, m) = 1$, that is, if no number bigger than one divides into n and m , we say that n and m are *relatively prime*. For positive integer n , the n^{th} order cyclotomic polynomial $Q_n(z)$ is defined⁴ to be

$$Q_n(z) = \prod_{1 \leq k < n, (k, n)=1} (z - r_n^k) \quad (11)$$

Because this may be an unfamiliar concept, we will work out a specific example, letting $n=15$. The numbers relatively prime to 15 between 1 and 15, are 1, 2, 4, 7, 8, 11, 13, and 14. Hence,

$$Q_{15}(z) = (z - r_{15}^1)(z - r_{15}^2)(z - r_{15}^4)(z - r_{15}^7)(z - r_{15}^8)(z - r_{15}^{11})(z - r_{15}^{13})(z - r_{15}^{14}) \quad (12)$$

Using convenient computational software such as Matlab, (12) may be expanded to give

$$Q_{15}(z) = z^8 - z^7 + z^5 - z^4 + z^3 - z + 1 \quad (13)$$

The roots of $Q_{15}(z)$ are the numbers r_{15}^k on the right side of (12). By their definition (9), they are all on the complex unit circle and this will clearly be true for the general $Q_n(z)$. Since the coefficient of every z in (12) is one, the coefficient of z^8 in (13) must be one; so $Q_{15}(z)$ is monic, as is every $Q_n(z)$. What is surprising, given that the numbers r_{15}^k are complex, is that all coefficients of $Q_{15}(z)$ are integers. It may be shown, using the methods of abstract algebra and number theory, that this is also true for the general $Q_n(z)$. Thus, each $Q_n(z)$ satisfies (i) and (ii) in Definition 1. We check part (iii) of the Definition 1 for $Q_{15}(z)$. By (13),

$$Q_{15}(z^2) = z^{16} - z^{14} + z^{10} - z^8 + z^6 - z^2 + 1 \quad (14)$$

and a computation shows

$$\frac{Q_{15}(z^2)}{Q_{15}(z)} = z^8 + z^7 - z^5 - z^4 - z^3 + z + 1 \quad (15)$$

The right side of (15) is a polynomial so that $Q_{15}(z)$ satisfies the Super Haar Condition. However, an examination of the polynomial on the right side of (23) shows its eight roots to be $r_{30}^1, r_{30}^7, r_{30}^{11}, r_{30}^{13}, r_{30}^{17}, r_{30}^{19}, r_{30}^{23}, r_{30}^{29}$

But the numbers that are relatively prime to 30 are 1, 7, 11, 13, 17, 19, 23, and 29, just the exponents in the list above. We therefore have

$$\frac{Q_{15}(z^2)}{Q_{15}(z)} = Q_{30}(z) \quad (16)$$

It may be shown that the relation indicated by (16) is true of the general $Q_n(z)$, if and only if n is odd. We state this result without proof:

Theorem 1: For the cyclotomic polynomials $Q_n(z)$

$$\frac{Q_n(z^2)}{Q_n(z)} = Q_{2n}(z) \quad (17)$$

if, and only if, n is odd.

The following is then immediate:

Corollary 1.1: The cyclotomic polynomial $Q_n(z)$ satisfies the Super Haar Condition if and only if n is odd.

The following shows that products of cyclotomic polynomials result in admissible scaling coefficients and the associated dilation coefficients can be easily computed.

Corollary 1.2: Suppose $n_1, n_2, \dots, n_k$ are odd positive integers, m any non-zero integer and let

$$S(z) = m Q_{n_1}(z) Q_{n_2}(z) \dots Q_{n_k}(z) \quad (18)$$

Then

$$D(z) = \frac{S(z^2)(z+1)}{2S(z)} = \frac{mQ_{n_1}(z^2)Q_{n_2}(z^2) \dots Q_{n_k}(z^2)(z+1)}{2mQ_{n_1}(z)Q_{n_2}(z) \dots Q_{n_k}(z)} = Q_{2n_1}(z)Q_{2n_2}(z) \dots Q_{2n_k}(z)(z+1)/2 \quad (19)$$

By (11) it may be shown that $z+1 = Q_2(z)$. Thus (19) may be written

$$D(z) = \frac{1}{2} Q_2(z)Q_{2n_1}(z)Q_{2n_2}(z) \dots Q_{2n_k}(z) \quad (20)$$

Now since each n_j on the right side of (20) is odd, each $2n_j$ is even but not divisible by four. Thus every subscript on the right side of (20) is even but not divisible by four.

It should be remarked that if we set $S_1(z) = (z+1)S(z)$, where $S(z)$ is given by (18), $S_1(z)$ is also Super Haar admissible. If $S_1(z)$ is substituted for $S(z)$ in (18) the resulting dilation coefficients are given by $D_1(z) = Q_{2n_1}(z)Q_{2n_2}(z) \dots Q_{2n_k}(z)(z^2+1)/2$, which, of course, is a polynomial so $S_1(z)$ is admissible. By (11) it may be shown that $z^2+1 = Q_4(z)$. Thus

$$D_1(z) = \frac{1}{2} Q_4(z)Q_{2n_1}(z)Q_{2n_2}(z) \dots Q_{2n_k}(z) \quad (21)$$

Since four is divisible by four, this does not precisely fit the pattern of (20). The anomaly is due, of course, to the factor $(z+1)$ in (19).

The proof to Theorem 1 is by elementary number theory and is not difficult. We do not give it here because in its present form it is rather long. Theorem 1 and its corollaries have implications for both theoretical and practical research. We discuss these in turn.

3. THEORETICAL RESEARCH INDICATED

3.1 Generalization to other types of wavelet

We have seen that cyclotomic polynomials are of value in the search for Super Haar admissible scaling coefficients and immediately provide the related scaling coefficients. Since these polynomials are fundamental to abstract algebra, they have been extensively studied and are known to satisfy many relationships similar to (19). (See Section 5.) Thus they may also be useful in the study of generalizations of Super Haar wavelets. For example, they may apply to Super extensions of wavelets other than Haar wavelets. For the application discussed in this paper, (4) is the defining equation. The factor $z+1$ in (4) results from the Haar scaling function. It might be possible to study another class of wavelet simply by changing this factor. On the other hand, the exponent of two in the factor $S(z^2)$ comes from the two in the factor $\phi(2t-k)$ on the right side of (3). If this factor was replaced by, say, $\phi(3t-k)$, as in ³, we would expect the equation corresponding to (4) would contain $S(z^3)$. A modification of the current theory would probably apply.

3.2 Necessary and sufficient conditions

As noted, the Super Haar Condition is sufficient but not necessary for the admissibility of the associated scaling conditions. We would like to find necessary and sufficient conditions. In this search, we believe the abstract algebraic approach is indicated. The following is known from abstract algebra ^{4,5}:

Theorem 2: Let $r_n^m = \exp(j2\pi \frac{m}{n})$ where n is a positive integer, $1 \leq m < n$ and $(m, n) = 1$. Then the unique monic polynomial with integer coefficients that cannot be factored and has r_n^m as a root is $Q_n(z)$

Thus the cyclotomic polynomials are, so to speak, the “building blocks” of polynomials with integer coefficients that have r_n^m as a root since every such polynomial is a product of cyclotomic polynomials. Now the rational numbers are defined to be all those numbers that can be written $\frac{m}{n}$, where n and m are integers. Thus Theorem 2 shows that the cyclotomic polynomials are the

building blocks of polynomials that have $\exp(j2\pi q)$ as a root, where q is a rational number. If z is a number on the complex unit circle, we must have either $z = \exp(j2\pi q)$ where q is a rational number or $z = \exp(j2\pi s)$ where s is an irrational number. We conjecture that *there is no polynomial* with integer coefficients that has $\exp(j2\pi s)$ as a root, where s is irrational. Proof of this is an essential step in the development of necessary conditions for our theory; that is, to the proof that essentially *all* Haar admissible scaling coefficients can be found by (18).

3.3 Characterization of polynomials

An m^{th} degree polynomial $P(z) = \sum_0^m p_k z^k$ is characterized by its $m+1$ coefficients p_k . If $P(z)$ is given in factored form, $P(z) = c \prod_1^m (z - z_i)$, it is characterized by its m roots z_i together with the constant c . Equation (11) defines cyclotomic polynomials in terms of their roots, r_n^k . As will become evident from the following, we need to be able to also characterize these polynomials in term of their coefficients.

The relationship between the roots and coefficients of any polynomial is known. Since the cyclotomic polynomials are monic, we suppose $P(z)$ is any monic polynomial and let

$$P(z) = \prod_1^m (z - z_i) = \sum_0^m p_k z^k \quad (22)$$

By expanding the middle expression above, it can be shown⁵ that

$$p_k = (-1)^{m-k} \sigma_{m-k} \text{ where } \sigma_r = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq m} z_{i_1} z_{i_2} \dots z_{i_r} \quad (23)$$

The expressions σ_r have been much studied and are called the *elementary symmetric polynomials*⁵.

We must add a little notation. From (11), the degree of the n^{th} order cyclotomic polynomial equals the number of positive integers less than n and relatively prime to n ; therefore, it is a function of n . In number theory, this function is referred to as the *totient* function of n . For convenience, when discussing cyclotomic polynomials, we will let $m(n)$ denote the totient function of n :

$$m(n) = \text{number of positive integers } k < n \text{ such that } (k, n) = 1 \text{ (totient function)} \quad (24)$$

We write (11) in a form like (22)

$$Q_n(z) = \prod_{1 \leq k < n, (k, n)=1} (z - r_n^k) = \sum_{s=0}^{m(n)} q_s z^s \quad (25)$$

From (22), (23), and (25) we have, for each n and $1 \leq r \leq m(n)$,

$$\sigma_r = \sum_{1 \leq s_1 < s_2 < \dots < s_r \leq m} r_n^{s_1 + s_2 + \dots + s_r}, \quad (s_i, n) = 1 \quad (26)$$

It is known that each coefficient q_s in (25) is an integer; and from (23) $q_s = (-1)^{m(n)-s} \sigma_{m(n)-s}$. Therefore every σ_r given by (26) is an integer. Using number theory, it should be possible to prove this directly from (26). The insights gained by doing this should be of value in characterizing the cyclotomic polynomials in terms of their coefficients.

3.4 Computation of total filter bank

A complete sub-band filter bank with decimation and upsampling is shown in Figure 1. The filters are identified as follows: $H_0(z)$ = lowpass analysis, $H_1(z)$ = highpass analysis, $G_0(z)$ = lowpass synthesis, $G_1(z)$ = highpass synthesis. (In this Section we use the results in ¹ and ² with slight modifications. For example, we write z^k rather than z^{-k} .)

Figure 1. Sub-band filter bank

In terms of our previous notation

$$H_0(z) = D(z) \quad (27)$$

where the possible values of $D(z)$ are given by (20) and (21). If we impose certain restrictions on our filter bank it is possible to find the complete bank from the known value of $D(z)$. We assume all filters are the same length $m+1$, where m is odd. Quadrature Mirror Filter perfect reconstruction will be obtained if

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2z^m \quad (28)$$

Aliasing is removed provided that

$$G_1(z) = H_0(-z) = D(-z) \quad (29)$$

$$H_1(z) = -G_0(-z) \quad (30)$$

Substituting (27), (29), and (30) into (28) results in

$$D(z)G_0(z) - D(-z)G_0(-z) = 2z^m \quad (31)$$

Defining $P(z) \equiv D(z)G_0(z)$, (31) becomes

$$P(z) - P(-z) = 2z^m \quad (32)$$

Now since $P(z)$ is a polynomial and for any integer k , $(z)^{2k} - (-z)^{2k} = 0$ but $(z)^{2k+1} - (-z)^{2k+1} = 2z^{2k+1}$, by (32) $P(z)$ must be of the form

$$P(z) \equiv D(z)G_0(z) = z^m + \sum_k p_{2k} z^{2k} \quad (33)$$

That is, the m^{th} odd coefficient of $P(z)$ must be one, all other odd coefficients must be zero, while its even coefficients may be anything. The coefficients of $D(z)$ are assumed to be known: $D(z) = \sum_0^m d_k z^k$; we set the unknown coefficients of $G_0(z)$ to $G_0(z) = \sum_0^m g_k z^k$ and we assume the coefficients of $G_0(z)$ are symmetric; that is $g_{m-k} = g_k$. Using these relations together with the fact that the m^{th} coefficient of $P(z) = D(z)G_0(z)$ must be one, we have

$$\sum_k d_k g_{m-k} = \sum_k d_k g_k = 1 \quad (34)$$

Since the other odd coefficients must be zero, for $n = 1, 2, 3, \dots, \frac{m-1}{2}$, with the convention that $g_{2n+k} = 0$ for $2n+k > m$, we have

$$\sum_k d_k g_{m-2n-k} = \sum_k d_k g_{2n+k} = 0 \quad (35)$$

Using (34) and (35), a matrix equation is obtained relating the coefficients of $D(z)$ with those of $G_0(z)$. A specific example is worked out in Section 6. In that example, it proved possible to solve for the coefficients of $G_0(z)$. However, a solution of (34), (35) for $G_0(z)$ might not always exist. For example, in ¹, two specific cases are worked out. In one case a unique solution is found whereas in the other case it is shown that a solution does not exist.

Due to (20) and (21), we now have a large class of admissible polynomials $D(z)$, and we understand at least some of the most important characteristics of these polynomials. It is hoped that using these notions, we may be able to determine which $D(z)$ will result in an admissible synthesis filter $G_0(z)$. It should be noted that a great many relationships hold in the cyclotomic polynomials in addition to the one we have used (17). A few of these are listed in Section 5. Using these, we may, for example, be able to characterize $G_0(z)$ as we have $D(z)$.

4. APPLIED RESEARCH INDICATED

In order for our theory to be of practical use in signal processing, two questions must be answered:

(i) Given a particular class of signals that we wish to process optimally according to some given criteria, how do we choose the scaling coefficients s_k (without regard to admissibility) to achieve this? That is, what would the ideal scaling coefficients be if we didn't have to worry about whether they are admissible?

(ii) How can we use (18) to find admissible scaling coefficients that match the optimal as closely as possible? (An associated problem is to define "closeness" mathematically.)

To give more meaning to (i) we discuss a general example. Suppose the signal processing we wish to do is denoising. That is, using Super Haar wavelets, we wish to recover a signal from data corrupted by noise. We assume the noise and signal have certain known characteristics. For example, the noise may be Gaussian white and the signal smooth. We may seek to recover a signal that is optimal with respect to, say, mean-squared error and smoothness ⁶. Equation (18) may be used in this investigation in two ways. First, (18) provides us with infinitely many admissible scaling function polynomials. The empirical approach would be to denoise our signal using a large number of these polynomials in a systematic manner and observe the results. The goal would be to abstract characteristics of the polynomials that give the best empirical results. Then, using (18) again, we could choose the polynomial $S(z)$ that most exhibits these characteristics. The second approach would be to deduce a priori from the given conditions the characteristics of the ideal polynomial. We would then use (18) to approximate this polynomial.

However, as noted in Section 3.3, at the present time in our theory, the cyclotomic polynomials are defined according to (11) only by their roots, not their coefficients. (Although, it may be shown that if only one root is exactly known the cyclotomic polynomial is determined and its coefficients easily generated.) But the empirical or theoretical investigation may provide the characteristics of the optimal polynomial as functions of, not its roots, but its coefficients. This could not directly lead us to the optimal admissible scaling coefficients, since they will be determined by taking a product of cyclotomic polynomials. These, in turn, are known only in terms of their roots.

This problem could possibly be solved if the relation between roots and coefficient discussed in Section 3.3 were better understood. Two other solutions suggest themselves. First, suppose we have the optimal scaling coefficients. We could find the roots of the associated polynomial by numerical calculation. We could then try to match these roots with cyclotomic polynomial roots. Second, we could try to find the characteristics of the optimal polynomial as a function of its roots rather than its scaling coefficients. Then we would again try to match these with cyclotomic polynomial roots.

5. A FEW RELATIONSHIPS SATISFIED BY CYCLOTOMIC POLYNOMIALS

For reference, we list the first twelve cyclotomic polynomials.

$$\begin{aligned} Q_1 &= z-1; & Q_2 &= z+1; & Q_3 &= z^2+z+1; & Q_4 &= z^2+1; & Q_5 &= z^4+z^3+z^2+z+1; & Q_6 &= z^2-z+1; \\ Q_7 &= z^6+z^5+z^4+z^3+z^2+z+1; & Q_8 &= z^4+1; & Q_9 &= z^6+z^3+1; & Q_{10} &= z^4-z^3+z^2-z+1; \\ Q_{11} &= z^{10}+z^9+z^8+z^7+z^6+z^5+z^4+z^3+z^2+z+1; & Q_{12} &= z^4-z^2+1 \end{aligned} \quad (36)$$

The following fundamental relationship is given in ⁴. For positive integer n

$$z^n - 1 = \prod_{k \text{ divides } n} Q_k(z) \quad (37)$$

Thus, for example, let $n = 4$; the number that divide 4 are 1, 2, and 4, so we should have

$$z^4 - 1 = Q_1(z)Q_2(z)Q_4(z) \quad (38)$$

and this may be verified by (36). We have also discovered the following relationships, which can be proved by elementary number theory. For positive integer n

$$Q_{2n+1}(z^2) = Q_{2n+1}(z)Q_{2n+1}(-z) \quad (39)$$

$$Q_{2n}(z^2) = Q_{4n}(z) \quad (40)$$

$$Q_{2n+1}(-z) = Q_{4n+2}(z) \quad (41)$$

If $m(n)$ = totient function evaluated at n then

$$z^{m(n)} Q_n(z^{-1}) = Q_n(z) \quad (42)$$

If p is prime,

$$Q_{p^n}(z) = Q_p(z^{p^{n-1}}) = \sum_{k=0}^{p-1} z^{kp^{n-1}} \quad (43)$$

6. A FILTER BANK EXAMPLE

In this example we set $m = k = 1$ and $n_1 = 15$ is (18) and calculate the filter bank shown in Figure 1. We thus have $S(z) = Q_{15}(z)$, which we have already found in equation (13):

$$Q_{15}(z) = z^8 - z^7 + z^5 - z^4 + z^3 - z + 1 \quad (13)$$

(Naturally $S(z)$ is identified with its coefficients: $S = [1, -1, 0, 1, -1, 1, 0, -1, 1]$.) Then since $n_1 = 15$ in (18), by (19),

$$D(z) = H_0(z) = \frac{1}{2} Q_{30}(z)(z+1) \quad (44)$$

$Q_{30}(z)$, given by (15), could be calculated from (11) using Matlab. However, it can also be found from (41):

$$Q_{30}(z) = Q_{15}(-z) = z^8 + z^7 - z^5 - z^4 - z^3 + z + 1 \quad (45)$$

Thus by (14),

$$H_0(z) = D(z) = \frac{1}{2} (z^9 + 2z^8 + z^7 - z^6 - 2z^5 - 2z^4 - z^3 + z^2 + 2z + 1) \quad (46)$$

That is $H_0 = \frac{1}{2} [1, 2, 1, -1, -2, -2, -1, 1, 2, 1]$. By (29) $G_1(z) = H_0(-z)$ so

$$G_1 = \frac{1}{2} [-1, 2, -1, -1, 2, -2, 1, 1, -2, 1] \quad (47)$$

Then G_0 may be found from (34) and (35), where d_k denotes the coefficients of $H_0 = D$ and g_k the coefficients of G_0 . Since the filters are of length 10, it would appear that (34) and (35) would lead to a 10 by 10 system of equations. However, if it is recalled that $d_{9-k} = d_k$, $g_{9-k} = g_k$ and $g_{2n+k} = 0$ for $2n+k > m$, a 5 by 5 system may be formed. This process is detailed in ¹. In this case the result is

$$\begin{aligned} d_1 g_9 + d_0 g_8 &= 0 \\ d_3 g_9 + d_2 g_8 + d_1 g_7 + d_0 g_6 &= 0 \\ d_4 g_9 + d_4 g_8 + d_3 g_7 + d_2 g_6 + (d_1 + d_0) g_5 &= 0 \\ d_2 g_9 + d_3 g_8 + (d_4 + d_0) g_7 + (d_4 + d_1) g_6 + (d_3 + d_2) g_5 &= 0 \\ d_0 g_9 + d_1 g_8 + d_2 g_7 + d_3 g_6 + d_4 g_5 &= 1 \end{aligned} \quad (48)$$

The unique solution to the above is

$$G_0 = [3, -6, 9, -9, 4, 4, -9, 9, -6, 3] \quad (49)$$

Since by (30) $H_1(z) = -G_0(-z)$,

$$H_1 = [3, 6, 9, 9, 4, -4, -9, -9, -6, -3] \quad (50)$$

Unfortunately, the Quadrature Mirror Filter given by (46), (47), (49) (50), while yielding perfect reproduction, did not appear to produce in any improvement in denoising over the ordinary Haar filter. More of the theoretic and applied research described in Sections 3 and 4 is needed to assess the value of cyclotomic polynomials in Super Haar and other Super filtering algorithms.

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