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REALIZING UNITS AS WHITEHEAD TORSIONS IN LOW DIMENSION

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1. INTRODUCTION

When the Whitehead torsion $\tau(f)$ of a homotopy equivalence $f: X \to Y$ between CW-complexes X and Y was invented (see [25], [26]), relatively little was known about the group $Wh\ \pi_1(Y)$ of equivalence classes of matrices in which $\tau(f)$ lies. The computation of $\tau(f)$ yields an invertible matrix N over $Z\pi_1(Y)$ (the integral group ring of the fundamental group of Y), but requires choices of bases in chain groups, and is meant to regard as trivial the elementary expansions and collapses of cells with "free faces"; so $\tau(f)$ is an equivalence class of matrices. Thus (writing π for $\pi_1(Y)$) let $GLZ\pi = \bigcup_n GL_n Z\pi$ with identifications:

$$M \longleftrightarrow \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$$

for each identity matrix I, and let $EZ\pi$ be the subgroup generated by matrices differing from I by one off-diagonal entry. Then $K_1Z\pi = GLZ\pi/EZ\pi$, and $Wh\ \pi$ is the cokernel of the composite:

$$\pm\pi \to GL_1Z\pi \to GLZ\pi \to K_1Z\pi \ .$$

Then $\tau(f)$ is the coset of N in $Wh\ \pi$. The homotopy equivalence f is called "simple" if $\tau(f) = 1$.

At first $Wh\ \pi$ was inaccessible to calculations for all but a very few groups π. For the last thirty years it has served as one of the seeds from which algebraic K-theory has grown. Finally algebraic techniques have begun to bring $Wh\ \pi$ within reach for computation with Whitehead torsions when π is a finite group. What follows is a sketch of the

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structure of $Wh\ \pi$, some recent results on the image of $GL_1 Z\pi$ in $K_1 Z\pi$, and a sample of applications to the simple homotopy classification of finite connected CW-complexes.

2. ALGEBRAIC BACKGROUND

For any ring R (with unit), $K_1 R$ is defined in the same way as $K_1 Z\pi$. Every standard determinant map defined on $GL_n R$ for every n factors through $GL_n R \to GLR \to K_1 R$. If R is commutative this includes the ordinary determinant; if R is a subring of a finite dimensional semi-simple Q-algebra, it includes the reduced norm. The kernel of the induced determinant on $K_1 R$ is denoted $SK_1 R$.

If π is finite, $K_1 Z\pi$ is a canonical (internal) direct product: $\{\pm 1\} \times \pi^{ab} \times SK_1 Z\pi \times F$, where $\pi^{ab} = \pi/[\pi,\pi]$ is the image of

$$\pi \to GL_1 Z\pi \to GLZ\pi \to K_1 Z\pi \quad ,$$

the group $SK_1 Z\pi$ is finite abelian, and F is a free abelian group of rank $r - q$ where r and q are the numbers of simple components in the real and rational group algebras of π. (For proofs see [2] and [23].)

For computations it is helpful to know for what values of n the composite $GL_n Z\pi \to GLZ\pi \to K_1 Z\pi$ is surjective. If π is infinite, n may have to be large (e.g., see [4, Theorem 7.3]); but if π is finite, then $GL_2 Z\pi \to K_1 Z\pi$ is surfective (see [3, Chapter V, §4]).

If π is abelian, $GL_1 Z\pi \to GLZ\pi \to K_1 Z\pi$ splits the exact sequence:

$$(1) \qquad 1 \to SK_1 Z\pi \to K_1 Z\pi \xrightarrow{\text{determinant}} GL_1 Z\pi \to 1 \quad ;$$

so $K_1 Z\pi$ is represented by units of $Z\pi$ if and only if $SK_1 Z\pi = 1$. If π is finite and nonabelian, $SK_1 Z\pi$ is the kernel of the reduced norm of $K_1 Z\pi$; but the reduced norm takes its values, not in $Z\pi$, but in the center of $Q\pi$. So the sequence (1) does not generalize. If π is finite and $Q\pi$ is Eichler, the reverse of the sequence (1) generalizes to an exact sequence:

$$(2) \qquad GL_1 Z\pi \to K_1 Z\pi \to SK_1 Z[\pi^{ab}] \to 1$$

(see [16]). So for finite groups π, $K_1 Z\pi$ is represented by units of $Z\pi$ provided the following condition holds:

(c_0) No quotient of π is included in the list of groups:
(a) binary polyhedral
(b) generalized quaternion
(c) $Z_{p^2} \times Z_{p^2}$ (p prime)
(d) $Z_p \times Z_p \times Z_p$, $Z_p \times Z_2 \times Z_2 \times Z_2$, or $Z_4 \times Z_2 \times Z_2$ (p odd prime)

The groups in (a) and (b) are the nonabelian finite subgroups of $GL_1\mathbb{H}$ where $\mathbb{H}$ is the division ring of real quaternions; avoidance of these as quotients of π is equivalent to the Eichler condition on $Q\pi$ (see [20, p. 344]). Avoidance of the quotients (c) and (d) is equivalent to the vanishing of $SK_1Z[\pi^{ab}]$ (see [1]).

EXAMPLES I. Among the finite groups satisfying condition (c_0) are all finite cyclic, dihedral and symmetric groups, as well as all finite simple groups.

3. STABILITY AND REALIZABILITY

Assume all CW-complexes are pointed, all maps preserve base points, and all homotopies are relative to base points. Let X be a finite connected CW-complex with fundamental group π. Let $E(X)$ denote the group of homotopy classes of homotopy self-equivalences of X. Simple homotopy classification is studied by means of the following device (see [7, p. 80]): Every CW-complex homotopy equivalent to X is simple homotopy equivalent to X if and only if Whitehead torsion $\tau: E(X) \to Wh\,\pi$ is surjective.

The size of representative matrices for $K_1Z\pi$ has a bearing on simple homotopy types:

THEOREM 1 (M. N. Dyer): If $GL_nZ\pi \to K_1Z\pi$ is surjective, then $\tau: E(X \vee nS^r) \to Wh\,\pi$ is surjective for all $r \geq 2$.

PROOF: See [10]. □

Therefore when π is finite, homotopy type and simple homotopy type of $X \vee 2S^r$ agree. When $K_1Z\pi$ is represented by units of $Z\pi$, they also agree for $X \vee S^r$.

4. TECHNIQUES IN DIMENSION TWO

A (π,2)-complex is a finite connected 2-dimensional CW-complex with fundamental group π. If $P = (g_1, \ldots, g_\sigma : r_1, \ldots, r_\tau)$ is a finite presentation of π and each relator r_j is a reduced word, the cellular model P of $\mathcal{P}$ is the $(\pi,2)$-complex with one 0-cell, an oriented 1-cell for each g_i, and a 2-cell for each r_j, attached to the 1-skeleton following the spelling of r_j. It is noted in [12] that every simple homotopy type of a $(\pi,2)$-complex contains the cellular model of such a presentation of π.

The graph whose vertices are homotopy types of $(\pi,2)$-complexes and whose edges connect the homotopy type of X to that of $X \vee S^2$ is a connected tree $HT(\pi,2)$, described in [12]. The Euler characteristic $\chi(X)$ introduces a notion of height on this tree. With the above notation (when π is finitely presented), $\chi(P) = 1 - \sigma + \tau$, which is determined by the deficiency $\sigma - \tau$ of the presentation $\mathcal{P}$.

If π is finite, the deficiency of every finite presentation is at most zero. Let $\mathrm{def}(\pi)$ denote the maximum deficiency for presentations of π. Then $1 - \mathrm{def}(\pi) \geq 1$ and is the minimum Euler characteristic χ_{min} in $HT(\pi,2)$. For any $(\pi,2)$-complex Y, $\chi(Y) - \chi_{min}$ is called the "level" of Y.

For finite π (by [6, Theorem 1.1]) there is exactly one homotopy type in $HT(\pi,2)$ at each level above zero. By Theorem 1, it follows that there is only one simple homotopy type at each level above one, and if π satisfies condition (c_0), there is only one simple homotopy type at level one as well.

To work at level zero, we must individually treat the cellular model of each presentation of π with maximum deficiency. If $\mathcal{P}$ (above) is such a presentation, the cellular chain complex of the universal cover $\tilde{P}$ of the cellular model P has the form:

$$\begin{array}{ccccc} C_2\tilde{P} & \xrightarrow{\partial_2} & C_1\tilde{P} & \xrightarrow{\partial_1} & C_0\tilde{P} \\ \| & & & & \| \\ (Z\pi)^{\tau} & & (Z\pi)^{\sigma} & & Z\pi \end{array}$$

where, representing these free $Z\pi$ modules as column vectors over a suitable choice of $Z\pi$-bases, ∂_2 is represented by the Jacobian matrix M of Fox derivatives $\partial r_j/\partial g_i$, and ∂_1 by the row $(g_i - 1)$ (see [12, pp. 35-36]). If N is in $GL_\tau Z\pi$ and $MN = M$, then the chain automorphism $(N, I_\sigma, 1)$ is realized by a homotopy self-equivalence of P with Whitehead torsion represented by N (see [12, Proposition 4]). To prove that simple homotopy type and homotopy type of P agree, we need only show that every element of $Wh\,\pi$ is represented by such a matrix N.

More generally, a (π,m)-complex is a finite connected m dimensional CW-complex X with $\pi_1(X) = \pi$ and $\pi_i(X) = 0$ for $1 < i < m$. If π is periodic with free period k, results obtained by the above technique lift to $(\pi, ik+2)$-complexes, for $i > 0$.

Indeed, since $\tilde{P}$ is simply connected,

$$C_2 \tilde{P} \to C_1 \tilde{P} \to C_0 \to \tilde{P} \to Z \xrightarrow{\varepsilon} 0$$

is exact, where ε is augmentation of $Z\pi$. If π has free period k, there is an exact extension:

$$0 \to Z \to F_{k-1} \to \cdots \to F_3 \to C_2 \tilde{P} \to C_1 \tilde{P} \to C_0 \tilde{P} \to Z \to 0,$$

where each F_j is a free $Z\pi$ module. Splice i copies of this sequence and $C_*\tilde{P}$ at the left to obtain the acyclic chain complex D_*:

$$C_2 \tilde{P} \to C_1 \tilde{P} \to C_0 \tilde{P} \to F_{ik-1} \to \cdots \to F_3$$
$$C_2 \tilde{P} \to C_1 \tilde{P} \to C_0 \tilde{P} \to Z \to 0 \ .$$

By [22, Theorem E], there is a $(\pi, ik+2)$-complex X with $C_*(X) = D_*$. The chain self-equivalence which is multiplication by N on D_{ik+2} and the identity elsewhere is geometrically realized by an element of $E(X)$ whose Whitehead torsion is represented by N. So homotopy type and simple homotopy type coincide for X.

5. METACYCLIC GROUPS OF DEFICIENCY MINUS ONE

For the next two sections fix the following notation. Let A and B be finite cyclic groups of orders m and s, generated by a and b respectively. A split metacyclic group π is defined by the split exact sequence of groups:

$$1 \to A \to \pi \to B \to 1.$$

It is the semidirect product $A \rtimes B$ where B acts on A by the splitting and conjugation. There is a positive integer α for which $b^{-1}ab = a^{\alpha}$, and m divides $\alpha^s - 1$. Let $P = (a,b : a^m, b^s, b^{-1}aba^{-\alpha})$ be the resulting presentation of π, with cellular model P. Also, if $x_1, \ldots, x_n$ are integers, let $(x_1, \ldots, x_n)$ denote their positive greatest common divisor.

The method of the previous section was used in [14] to prove the following:

THEOREM 2: The image of $\tau : E(P) \to \mathrm{Wh}\,\pi$ contains all elements represented by units of $Z\pi$. □

So P is a root at level zero of $HT(\pi,2)$ and has simple homotopy type equal to homotopy type, provided the following three conditions hold:

(c_1) $\mathrm{def}(\pi) \neq 0$,

(c_2) $SK_1Z[\pi_{ab}] = 1$,

(c_3) $Q\pi$ is Eichler.

In [24], J. W. Wamsley determined $\mathrm{def}(\pi)$. Using his criterion we can re-express each condition in terms of P:

(c_1') $(m,\ \alpha - 1,\ (\alpha^s - 1)/m,\ (\alpha^s - 1)/(\alpha - 1)) \neq 1$,

(c_2') $(m,\ \alpha - 1,\ s)$ is square free,

(c_3') If 4 divides s, then $(m,\ \alpha + 1) \leq 2$,

or group theoretically:

(c_1'') For some prime p, $Z_p \times Z_p$ is a quotient of π, and the action of B on $A \equiv Z_m$ extends to an action of Z_{pm}.

(c_2'') There is no prime q for which $Z_{q^2} \times Z_{q^2}$ is a quotient of π,

(c_3'') No generalized quaternion group is a quotient of π.

To construct examples, choose positive integers m and s with a common prime factor p, and choose a positive integer $\alpha \equiv 1 \pmod p$ with $\alpha^s \equiv 1 \pmod{pm}$ and satisfying (c_2') and (c_3').

EXAMPLES II: If $s = 2$ this yields the dihedral groups of order $2m$ (m even) and other groups such as $(a, b : a^{12}, b^2, b^{-1}aba^{-5})$. If p is an odd prime we also get examples such as $(a, b : a^{p^2}, b^{p^2}, b^{-1}aba^{-(p+1)})$, for which $SK_1Z\pi \neq 1$ (see [17, Theorem 3]).

When ρ is a finite abelian group, simple homotopy type and homotopy type coincide for $(\rho,2)$-complexes provided $K_1Z\rho$ is represented by units (see [21] and [5]), and this condition coincides with the vanishing of $SK_1Z\rho$. The last example above shows that vanishing of SK_1 is not a necessary condition when ρ is nonabelian.

6. METACYCLIC GROUPS OF DEFICIENCY ZERO

Suppose now that $\mathrm{def}(\pi) = 0$, so π has a presentation $\mathcal{R}$ with the same number of generators as relators. Knowing the Fox Jacobian M of $\mathcal{R}$, we look for matrices N representing $Wh\,\pi$ with $MN = M$. If units represent $Wh\,\pi$, we may need unit representatives which take a particular form when expressed in terms of the generators in $\mathcal{R}$. By this method, the simple homotopy classification was completed (in [14]) for $(\pi,2)$-complexes, where π is dihedral of order $2m$ (m odd). We present a slight generalization here.

LEMMA 3: If s is even and π (defined in the last section) is presented by $\mathcal{R} = (a, b : a^m b^{-s}, r)$ with the same generators as in $\mathcal{P}$ and some reduced word r, then $GL_1 Z\pi \cap (1 + (1 - b + b^2 - \cdots - b^{s-1})ZA)$ lies in the image of $\tau: E(R) \to Wh\,\pi$, where R is the cellular model of $\mathcal{R}$.

PROOF: The Fox Jacobian of $\mathcal{R}$ has the form:

$$M = \begin{bmatrix} 1 + a + \cdots + a^{m-1} & * \\ -(1 + b + \cdots + b^{s-1}) & * \end{bmatrix}$$

Suppose $u = 1 + (1 - b + b^2 - \cdots - b^{s-1})p(a)$ is in $GL_1 Z\pi$ where $p(x)$ is in $Z[x]$. Let $\sigma: Z\pi \to ZB$ be the partial augmentation defined by $\sigma(a) = 1$. If

$$N = \begin{bmatrix} u\sigma(u)^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

then $MN = M$; so $u\sigma(u)^{-1}$ represents a Whitehead torsion from $E(R)$.

If s is even there is a homomorphism $\mu: ZB \to Z$ taking b to -1. Then $1 + sp(1) = \mu\sigma(u) \in GL_1 Z = \{\pm 1\}$; so either $p(1) = 0$ (whence $\sigma(u) = 1$) or $s = 2$ and $p(1) = -1$ (whence $\sigma(u) = b$). In either case, $u\sigma(u)^{-1} \equiv u$ in $Wh\,\pi$. □

EXAMPLES III. In [24] J. W. Wamsley pointed out that if $(m, \alpha - 1) = 1$, then π is presented by $\mathcal{R} = (a, b : a^m b^{-s}, b^{-1}a^t b a^{-t-1})$ where $t(\alpha - 1) \equiv 1 \pmod m$. Or if $(\alpha - 1, (\alpha^s - 1)/m) = 1$, then π has a presentation $\mathcal{R} = (a, b : a^m b^{-s}, b^{-1}aba^{-\alpha})$. In both cases a and b are the same generators used in $\mathcal{P}$.

In [14, Theorem 6] we found restrictions on π so that units like those in Lemma 3 represent most of $K_1Z\pi$. Following R. Oliver's recent work on $SK_1Z\rho$ for finite groups ρ (see [17], [18] and [19]), we can make the hypotheses more explicit:

THEOREM 4: Suppose π is a finite split metacyclic group (as at the beginning of §5) and satisfies:

(c_4) $(m, \alpha - 1, m/(m, \alpha - 1)) = 1$,
(c_5) The multiplicative order of $\alpha \pmod p$ is s for any prime p dividing $m/(m, \alpha - 1)$,
(c_6) $(m, \alpha - 1, s)$ is square free,

or their group theoretic counterparts:

(c_4') $\pi \to \pi^{ab}$ splits,
(c_5') The action of B on A induces either a faithful or trivial action on each quotient of A,
(c_6') There is no prime q for which $Z_{q^2} \times Z_{q^2}$ is a quotient of π.

Then $K_1Z\pi$ is represented by $GL_1Z[\pi^{ab}]$ together with $GL_1Z\pi \cap (1 + b + b^2 + \cdots + b^{s-1})ZA)$.

PROOF: The hypotheses are the same as those in [14, Theorem 6] except that (c_6) replaces the requirement $SK_1Z\pi = 1$. Notice that (c_6) is the same as (c_2') of §5, which says $SK_1Z[\pi^{ab}] = 1$. But by [15, Theorem 1], there is a surjection $SK_1Z\pi \to SK_1Z[\pi^{ab}]$; and by induction on elementary subgroups [19, Theorem 3], if $SK_1Z[\pi^{ab}] = 1$, then $SK_1Z\pi = 1$ in this example. □

COROLLARY 5: If, in addition, s is even and π^{ab} has exponent dividing 4 or 6, then $Wh\ \pi$ is represented by elements in $GL_1Z\pi \cap (1 + (1 - b + b^2 - \cdots - b^{s-1})ZA)$.

PROOF: In the proof of [14, Theorem 6] we define a norm $\Delta\rho$ on $K_1Z\pi$ whose kernel is represented by $GL_1Z[\pi^{ab}]$. Direct calculation shows that $\Delta\rho(1 + (1 - b + b^2 - \cdots - b^{s-1})p(a)) = \Delta\rho(1 + (1 + b + b^2 + \cdots + b^{s-1})p(a))$ if s is even. By [13, Theorem 11], $GL_1Z[\pi^{ab}] = \pm\pi^{ab}$ exactly when the exponent of π^{ab} divides 4 or 6. □

If both Lemma 3 and Corollary 5 apply to π we get a $(\pi,2)$-complex R with $\chi(R) = 1$ (hence at level zero) for which simple homotopy type and homotopy type coincide.

EXAMPLES IV: Combining the hypotheses of Theorem 4 and Corollary 5 with the condition $(m, \alpha - 1) = 1$ required for the first presentation R in Examples III, we are left with the following groups π:

1. m is odd, $s = 2$, $\alpha = m - 1$; these are dihedral of order $2m$ (m odd).
2. m is a product of primes $\equiv 1 \pmod 4$, $s = 4$, and α is chosen so that $(m, \alpha^2 - 1) = 1$ and m divides $\alpha^2 + 1$.
3. m is a product of primes $\equiv 1 \pmod 6$, $s = 6$, and α is chosen so that $(m, \alpha^4 + \alpha^3 - \alpha + 1) = 1$ and m divides $\alpha^2 - \alpha + 1$.

The hypotheses of Theorem 4 and Corollary 5 with the condition $(\alpha - 1, (\alpha^s - 1)/m) = 1$ for the second presentation R in Examples III leave us with exactly one other group: $m = 15$, $s = 2$, $\alpha = 4$.

7. CONCLUSIONS

To summarize, if X is a finite connected CW-complex with finite fundamental group π, then homotopy type and simple homotopy type coincide for $X \vee S^2$ (and is determined by the Euler characteristic) if π satisfies condition (c_0). Suppose further that X has dimension two and π is split metacyclic. Then $HT(\pi,2)$ has a (level zero) root with simple homotopy type equal to homotopy type, provided π either satisfies conditions (c_1), (c_2) and (c_3), or is among the groups listed in Examples IV.

If π is finite, periodic with free period four, and $Q\pi$ satisfies the Eichler condition, then $HT(\pi,2)$ is a "stalk": it has only one homotopy type at each level (see [8, Theorem 10c]). Among the groups mentioned above, this holds for π dihedral of order $2m$ (m odd) and for the split metacyclic group with $m = 15$, $s = 2$, $\alpha = 4$; so for these groups, simple homotopy type is determined by the Euler characteristic. Such strong results for the other metacyclic groups listed above await the computation of the roots of their homotopy trees $HT(\pi,2)$. For a survey of this problem, see [11].

Although these results may appear fragmentary, it is only because they rely on an intersection of results on deficiency of finitely presented groups, homotopy theory of $(\pi,2)$-complexes, representation of $Wh\,\pi$ by units, and a trial and error matching of Fox Jacobians with matrix representatives of Wh π. It seems likely that this intersection will grow, and that some more elegant results will emerge.

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