

COMBINATORIAL EQUIVALENCE BETWEEN GROUP PRESENTATIONS

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ABSTRACT. Let

$$\mathcal{P} = (x_1, \dots, x_n : W_1, \dots, W_m) \quad \text{and} \quad \mathcal{R} = (x_1, \dots, x_n : R_1, \dots, R_m)$$

be two presentations, with the same generators, for a group π . In this note, we give a necessary and sufficient criterion which insures the existence of a combinatorial equivalence between \mathcal{P} and \mathcal{R} requiring only replacement operations.

I. Introduction. Let

$$(1) \quad \mathcal{P} = (x_1, \dots, x_n : W_1, \dots, W_m)$$

and

$$(2) \quad \mathcal{R} = (x_1, \dots, x_n : R_1, \dots, R_m)$$

be two presentations, with the same generators, for a group π (i.e., the normal closures of W_1, \dots, W_m and of R_1, \dots, R_m in the free group F on x_1, \dots, x_n coincide). In [8, p. 95], A. J. Sieradski makes the following conjecture: Suppose there exists a combinatorial equivalence between \mathcal{P} and \mathcal{R} inducing the identity isomorphism. Then there exists a combinatorial equivalence between \mathcal{P} and \mathcal{R} requiring only replacement operations. We are unable to prove this conjecture; however, we give a necessary and sufficient criterion which insures its existence.

II. Definition of combinatorial equivalence. Let π be any group, and let

$$\mathcal{R} = (x_1, \dots, x_n : R_1, \dots, R_m) = (x_i : R_j)$$

be a finite presentation for π . We have the following *combinatorial operations* on the presentation \mathcal{R} :

(1) *The expansion operation.* It appends to the presentation \mathcal{R} a new generator $x \notin \{x_i\}$ and a relator Wx^{-1} where W is an element in the free group F on generators x_1, \dots, x_n . Therefore

$$(x_i : R_j) \rightarrow (x_i, x : R_j, Wx^{-1}).$$

(2) *The contraction operation.* It simultaneously deletes a generator x and a relator Wx^{-1} provided that W and the remaining relators R_j are elements in the free group F generated by the remaining generators x_1, \dots, x_n . Therefore $(x_i, x : R_j, Wx^{-1}) \rightarrow (x_i : R_j)$.

(3) *The replacement operation.* It replaces a single relator R by a new relator S provided that S and $R^{\pm 1}$ are conjugates modulo the normal subgroup $N(R_j)$ of F

Received by the editors June 10, 1981.

1980 *Mathematics Subject Classification.* Primary 20E06.

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0002-9939/82/0000-0072/\$01.75

generated by the other relators R_j which are kept unchanged. Hence $(x_i: R_j, R) \rightarrow (x_i: R_j, S)$.

Let \mathcal{P} and \mathcal{R} be two presentations for the same group π . We say \mathcal{P} and \mathcal{R} are *combinatorially equivalent* if there is a sequence

$$\mathcal{P} = \mathcal{P}_0 \rightarrow \mathcal{P}_1 \rightarrow \cdots \rightarrow \mathcal{P}_{s-1} \rightarrow \mathcal{P}_s = \mathcal{R}$$

of combinatorial operations which begin with \mathcal{P} and end with \mathcal{R} .

Note that we do not permit the Tietze operation of adding trivial relators to the presentation. This restriction is motivated by topological considerations. We urge the reader to read [8] for details.

III. Necessary and sufficient criterion. Suppose the presentations (1) and (2), with the same deficiency and the same generators, present the same group π . We will now formulate the necessary and sufficient conditions.

Let F denote the free group generated by x_1, \dots, x_n and R the normal closure in F of R_1, \dots, R_m . Let \bar{F} denote the free group on symbols r_1, \dots, r_m . Since each $W_i \in R$, we can write

$$W_i = \prod_{k=1}^{m_i} (x_{ik} R_{ik} x_{ik}^{-1})^{\epsilon_{ik}}$$

where $R_{ik} = R_j$ for some j , $1 \leq j \leq m$, $x_{ik} \in F$, and $\epsilon_{ik} = \pm 1$. Let

$$w_i = \prod_{k=1}^{m_i} (x_{ik} r_{ik} x_{ik}^{-1})^{\epsilon_{ik}}$$

where if $R_{ik} = R_j$ in W_i , then $r_{ik} = r_j$ in w_i .

Let J denote the $m \times m$ matrix

$$J = \begin{vmatrix} \frac{\partial w_1}{\partial r_1} & \cdots & \frac{\partial w_1}{\partial r_m} \\ \vdots & & \vdots \\ \frac{\partial w_m}{\partial r_1} & \cdots & \frac{\partial w_m}{\partial r_m} \end{vmatrix}$$

where $\partial/\partial r_j: ZF \rightarrow ZF$ denotes the j th free partial derivative [2].

THEOREM. *Notation as above. Then the following are equivalent:*

- (1) *There is a combinatorial equivalence between \mathcal{P} and \mathcal{R} requiring only replacement operations.*
- (2) *$\{x_1, \dots, x_n, w_1, \dots, w_m\}$ forms a generating set for the free product $F * \bar{F}$.*
- (3) *J has a right inverse.*

PROOF. (1) \Rightarrow (2) Suppose \mathcal{P} is combinatorially equivalent to \mathcal{R} using only replacement operations. Now all replacement operations are composites of these three replacement operations [8, p. 69]:

- (a) Replace a single relator R_i by its inverse R_i^{-1} .
- (b) Replace a relator R_i by the product $R_i R_j$ or $R_j R_i$ with a different relator R_j .
- (c) Replace a relator R_i by $x^\epsilon R_i x^{-\epsilon}$ by a generator x , $\epsilon = \pm 1$.

This means that we can convert $x_1, \dots, x_n, r_1, \dots, r_m$ into $x_1, \dots, x_n, w_1, \dots, w_m$ using transformations which leave x_1, \dots, x_n fixed and on the remaining generators the following operations:

(a') Replace r by r' .

(b') Replace r by $r\bar{r}$ or $\bar{r}r$ where \bar{r} is a different generator.

(c') Replace r by $x^\epsilon r x^{-\epsilon}$ where $x = x_i, 1 \leq i \leq n$.

But these are the elementary Nielsen transformations [3, p. 130]. Hence $x_1, \dots, x_n, w_1, \dots, w_m$ forms a generating set for $F * \bar{F}$.

(2) \Rightarrow (1) Suppose $\{x_1, \dots, x_n, w_1, \dots, w_m\}$ generates $F * \bar{F}$. Then by a result of Rapaport [7, §5], it follows that one can obtain $x_1, \dots, x_n, w_1, \dots, w_m$ from $x_1, \dots, x_n, r_1, \dots, r_m$ using transformations which leave x_1, \dots, x_n fixed and on the remaining generators the operations (a'), (b'), and (c') described above. But then one can construct a combinatorial equivalence between \mathcal{P} and \mathcal{R} using only replacement operations.

(2) \Rightarrow (3) Suppose $\{x_1, \dots, x_n, w_1, \dots, w_m\}$ forms a generating set for $F * \bar{F}$. Then by the Inverse Function Theorem [1, p. 635], the Jacobian which has the form

$$\begin{vmatrix} I_n & 0 \\ A & J \end{vmatrix}$$

has a right inverse B . By a result of Kaplansky [6], B is a two-sided inverse. Therefore J has a right inverse.

(3) \Rightarrow (2) Suppose J has a right inverse H . Then the Jacobian which has the form

$$\begin{vmatrix} I_n & 0_m \\ A_{m \times n} & J_m \end{vmatrix}$$

has a right inverse

$$\begin{vmatrix} I_n & 0_m \\ -H_m A_{m \times n} & H_m \end{vmatrix}.$$

The result now follows from the Inverse Function Theorem [1, p. 635]. This completes the proof.

EXAMPLE. Consider the two presentations

$$\mathcal{P} = (x, y, z: W_1 = x^5, W_2 = y^5, W_3 = z^5, W_4 = [x, y], W_5 = [x, z], W_6 = [y, z])$$

and

$$\mathcal{R} = (x, y, z: R_1 = x^5, R_2 = y^5, R_3 = z^5, R_4 = [x^4, y], R_5 = [x, z], R_6 = [y, z])$$

of the group $Z_5 \times Z_5 \times Z_5$. We can write

$$W_4 = xyR_1^{-1}y^{-1}R_4^{-1}R_1x^{-1}.$$

Therefore, $w_4 = xyr^{-1}y^{-1}r_4^{-1}r_1x^{-1}$.

Clearly $\{x, y, z, w_1, w_2, w_3, w_4, w_5\}$ forms a generating set for $F * \bar{F}$ and the matrix J is invertible. Thus there is a combinatorial equivalence between \mathcal{P} and \mathcal{R} using only replacement operations.

REMARK. In [4 and 5], examples are given of presentations which are combinatorially equivalent, but there does not exist a combinatorial equivalence between them requiring replacement operations only; however, these presentations do not have the same generators.

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