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SPACES DOMINATED BY TWO-COMPLEXES

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ABSTRACT. We say a group G is of geometric dimension $\leqslant 2$ if there is an aspherical 2-dimensional CW-complex P with fundamental group isomorphic to G. In this note, we study the following problem: Suppose G is a group of geometric dimension $\leqslant 2$ with associated aspherical 2-dimensional CW-complex P. Suppose further that X is a CW-complex having fundamental group isomorphic to G and that X is dominated by a 2-complex. If the Wall invariant $\mathrm{Wa}_2[X] \in \tilde{K}_0(ZG)$ vanishes, does X have the same homotopy type as $P \vee kS^2$ where kS^2 denotes the sum of k copies of the 2-sphere S^2?

Throughout this paper, all CW-complexes are finite connected with some zero cell chosen as base point, all maps preserve base points, and all homotopies are relative to base points.

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The answer to this question, in general, is no. In fact, Dunwoody [1] has constructed a 2-complex X which is not homotopy equivalent to $P \vee S^2$ where P denotes the geometric realization of the presentation $(a, b\colon a^2b^{-3})$ for the trefoil group.

LEMMA. *Let G, P and X be as above. Then π_2X as a ZG-module is stably free, i.e., $\pi_2X \oplus ZG^k \simeq ZG^l$ for some k and l.*

PROOF. Because the Wall invariant of X is zero, we may assume that X is a finite 3-complex [7, Theorem F, p. 66]. By [2, Theorem 4.1, p. 236] or [8, Theorem 1, p. 409], for some m, $X \vee mS^2$ is homotopy equivalent to Z where Z is a 2-complex with $\pi_1Z \simeq G$. Since Z and P are both 2-complexes with isomorphic fundamental groups, by a well-known result of J. H. C. Whitehead, $Z \vee tS^2 \simeq P \vee lS^2$ for some t and l. Therefore, $X \vee (m + t)S^2 \simeq Z \vee tS^2 \simeq P \vee lS^2$ so that $\pi_2X \oplus ZG^k \simeq \pi_2P \oplus ZG^l$ (where $k = m + t$). But P is aspherical, therefore $\pi_2P = 0$, hence $\pi_2X \oplus ZG^k \simeq ZG^l$.

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THEOREM 1. *Let X and P be as above. If Euler characters $\chi(X) = \chi(P)$, then X and P have the same homotopy type.*

PROOF. Since $\chi(X) = \chi(P)$, we must have $k = l$ and $\pi_2 X \oplus ZG^k = ZG^k$. By a result of Kaplansky [6], this means that $\pi_2 X = 0$. Because X is dominated by a 2-complex Y, the universal cover $\tilde{X}$ is dominated by a 2-complex $\tilde{Y}$ (this follows from the unique homotopy lifting property and the fact that we are working in the based category). Thus we have the commutative diagram

$$\begin{array}{ccc} H_i\tilde{Y} & \rightarrow & H_i\tilde{X} \\ \nwarrow & & \swarrow \text{id} \\ & H_i\tilde{X} & \end{array}$$

This means that the map $H_i\tilde{Y} \to H_i\tilde{X}$ is onto. Since $H_i\tilde{Y} = 0$ for $i \geqslant 3$, $H_i\tilde{X} = 0$ for $i \geqslant 3$. Therefore $\tilde{X} \simeq *$, and so X is aspherical. The result now follows.

THEOREM 2. *Let X be a CW-complex with fundamental group free abelian of rank two. If X is dominated by a 2-complex, then X is homotopy equivalent to $P \vee tS^2$ where P is the torus (i.e., geometric realization of the presentation $(a, b\colon aba^{-1}b^{-1})$ of $Z \times Z$).*

PROOF. For any finitely generated free abelian group H, all finitely generated projective ZH-modules are free. Therefore, $\tilde{K}_0(Z \times Z) = 0$ and, by the Lemma above, $\pi_2 X \simeq ZG^{l-k}$. Now it is possible to construct a homotopy equivalence between X and $P \vee (l-k)S^2$ using [5, Theorem 3, p. 26].

EXAMPLE. Let X denote the complement $S^4 - k(S^2)$ of the spun trefoil knot $(S^4, k(S^2))$ where $k(S^2) \subset S^4$ is a 2-sphere formed by spinning the trefoil knot about the standard 2-sphere S^2. Then we can obtain a 3-complex K which is a deformation retract of X and such that the Euler character $\chi(K) = 0$ (see [4] for details). Because $\pi_2 X \neq 0$, in view of Theorem 1, K cannot be dominated by a 2-complex.

In the Lemma above, we saw that $\pi_2 X \oplus ZG^k \simeq ZG^l$. This implies that $X \vee kS^2 \simeq P \vee lS^2$. In order that $X \simeq P \vee (l-k)S^2$, we must have that $\pi_2 X$ is actually free. Unfortunately, $\pi_2 X$ need not be free. The 2-complex X in Dunwoody's example (see above) is such that $\pi_2 X \oplus ZG \simeq ZG^2$, yet $\pi_2 X$ is not isomorphic to ZG. We give below a criterion which insures the freeness of this module in the case when $l \geqslant 2k$.

THEOREM 3. *Let G, P, and X be as above with $l \geqslant 2k$. Then $X \simeq P \vee (l-k)S^2$ if and only if there is a homotopy equivalence $f\colon X \vee kS^2 \to P \vee lS^2$, which on the second homotopy group induces the projection map, i.e., the following diagram*

$$\begin{array}{ccc} \pi_2(X \vee kS^2) & \xrightarrow{f_*} & \pi_2(P \vee lS^2) \\ \| & & \| \\ \pi_2 X \oplus ZG^k & & ZG^l \\ \downarrow \text{projection} & & \| \\ ZG^k & \subset & ZG^k \oplus ZG^{l-k} \end{array}$$

commutes such that there is a ZG-basis $\{d_1, \ldots, d_l\}$ of $\pi_2 X \oplus ZG^k$ $(\simeq ZG^l)$ such that for some $s \leqslant l-k$, the images $\{f_* d_1, \ldots, f_* d_s\}$ generate $ZG^k \subset ZG^l$.

PROOF. If $X \simeq P \vee (l-k)S^2$, then $\pi_2 X$ is isomorphic to ZG^{l-k}. Hence it is possible to construct a desired homotopy equivalence f using Theorem 3, p. 236 of [5]. Conversely, suppose there is a homotopy equivalence $f\colon X \vee kS^2 \to P \vee lS^2$ having the property stated in the theorem. By Gabel's Lemma [3, p. 39], it follows that $\pi_2 X \simeq ZG^{l-k}$. Hence $X \simeq P \vee (l-k)S^2$.

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