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SURJECTIVE STABILITY OF UNITS AND SIMPLE HOMOTOPY TYPE

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1. Introduction

Recently M.N. Dyer has shown that the classification of CW-complexes with fundamental group G into simple homotopy types is related to the size of matrices over ZG needed to represent all elements of the Whitehead group of G (see [9, Theorem 4]). For other purposes O.S. Rothaus (in [18]) has displayed units representing the Whitehead group of dihedral groups of order $2p$, for odd primes p. In this paper, Rothaus' result is generalized and applied to simple homotopy classification.

We begin in the next section with a general discussion of the representation of K_1ZG by units of ZG. The two sections which follow describe alternative norms and their connection with the reduced norm. The units representing K_1ZG for certain metacyclic groups G are then given. The last section is devoted to topological applications.

The following notation, conventions and definitions are used. Rings R are associative with 1. The center of R is $\mathscr{Z}(R)$. The group of units in R is $\mathrm{GL}_1(R) = R^*$. Elementary $n \times n$ matrices are those in the subgroup $\mathrm{E}_n(R)$ of $\mathrm{GL}_n(R)$ generated by matrices which equal the identity matrix except possibly at a single off-diagonal entry. The stabilization maps: $M_n(R) \to M_{n+r}(R)$ take a matrix M to

$$\begin{bmatrix} M & 0 \\ 0 & I_r \end{bmatrix}.$$

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If matrices connected by stabilization are identified, the unions $\bigcup_n \mathrm{GL}_n(R)$ and $\bigcup_n \mathrm{E}_n(R)$ become GLR and ER respectively, and K_1R is the quotient GLR/ER.

If R is a finite dimensional central simple algebra over a field F, and K is a commutative ring extension of F for which there is a K-algebra isomorphism $\beta: K \otimes_F R \cong M_n(K)$, the reduced norm $\nu: R \to F$ is defined by $\nu(r) = \det \beta(1 \otimes r)$, where det means determinant. (Such a K always exists; a splitting field of R will do. The map ν is independent of the choice of K and β.) For G a finite group, the reduced norm $\nu: QG \to Z(QG)$ is defined to be the reduced norm on every simple component of QG.

2. Stability of units

If G is a finite group, H. Bass (in [1, Chapter V, §4]) and L.N. Vaserstein (in [20]) have shown that the map induced by stabilization, $s_n: \mathrm{GL}_n ZG/\mathrm{E}_n ZG \to K_1 ZG$, is surjective for $n \geq 2$ and injective for $n \geq 3$. As a consequence, if we allow only the elementary row operation of adding a multiple of one row to another, then for $n \geq 3$, every matrix in $\mathrm{GL}_n(ZG)$ is row equivalent to one of the form

$$\left[\begin{array}{cc|c} * & * & \\ * & * & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right].$$

Which elements of $\mathrm{GL}_n(ZG)$ are actually row equivalent to a matrix

$$\left[\begin{array}{c|c} * & 0 \\ \hline 0 & I_{n-1} \end{array}\right]?$$

Equivalently, what is the image $U = s_1(ZG^*)$ in $K_1 ZG$?

When G is abelian, the determinant induces a surjective map, $K_1 ZG \to ZG^*$, with kernel denoted by $SK_1 ZG$. Since $\det s_1$ is the identity map on ZG^*, there is an internal direct product decomposition: $K_1 ZG = SK_1 ZG \times U$. The finite abelian groups G for which $SK_1 KG = 1$ have been almost completely determined (see [19]).

When G is nonabelian, the determinant is generalized to the reduced norm, ν, which takes $\mathrm{GL}_n ZG$ into the units of the integral closure of Z in the center of QG. The map induced on $K_1 ZG$ is also called the reduced norm ν, with kernel $SK_1 ZG$. The commutative case and, so far, all noncommunicative computations suggest that $K_1 ZG = SK_1 ZG \times U$ for any finite group G. Accordingly we propose two conjectures:

Conjecture 1. On ZG^*, ν and s_1 have the same kernel; that is, $SK_1 ZG \cap U = 1$.

Conjecture 2. Under ν, $GL_2 ZG$ and ZG^* have the same image; that is, $SK_1 ZG \cdot U = K_1 ZG$.

If the conjectures hold, then K_1ZG/U is SK_1ZG, which is subject to the cyclic and hyperelementary induction theorems developed in [12, pp. 122–23]. There appears to be no direct construction of a G_0ZG Frobenius module structure on K_1ZG/U, but a direct argument provides some induction tools:

Theorem 3. *Let $\mathscr{C}$ and $\mathscr{H}$ denote the sets of cyclic and hyperelementary subgroups of a finite group G, respectively.*

(a) *If $A(G)$ is the Artin exponent of G (see [13]), then $K_1ZG/s_1(\prod_{C\in\mathscr{C}} ZC^*)$ has exponent dividing $A(G)^2$.*

(b) *If $K_1ZH = s_1(ZH^*)$ for every H in $\mathscr{H}$, then $K_1ZG = s_1(\prod_{H\in\mathscr{H}} ZH^*)$.*

Proof. Let $i.: K_1ZS \to K_1ZG$ denote the map induces by inclusion of a subgroup S into G. By T.Y. Lam's extension of the Artin and Berman-Witt induction theorems, $K_1ZG/\prod_{C\in\mathscr{C}} i.K_1ZC$ has exponent dividing $A(G)^2$, and $K_1ZG = \prod_{H\in\mathscr{H}} i.K_1ZH$. Since $SK_1ZC = 1$ for any finite cyclic group C, $K_1ZC = s_1(ZC^*)$. By hypothesis, $K_1ZH = s_1(ZH^*)$ for all H in $\mathscr{H}$. Finally, s_1 is a natural transformation from $Z[-]^*$ to $K_1Z[-]$; so $i.s_1(ZS^*)$ is ZS^* stabilized into K_1ZG. □

In this paper computations are completed only for groups G with $SK_1ZG = 1$. For these, Conjecture 1 is automatic and Conjecture 2 reduces to the surjectivity of s_1.

3. Norms

If R is a ring, S is a commutative ring and $\rho: R \to M_m(S)$ is a ring homomorphism, then so is the entrywise map $\rho: M_n(R) \to M_{nm}(S)$. Following ρ by the determinant defines a multiplicative map, $\det \rho$, taking I_n to 1 (in S); so it restricts to a group homomorphism, $\mathrm{GL}_n(R) \to S^*$, which is compatible with stabilization and kills elementary matrices. Therefore ρ determines a norm $\Delta\rho: K_1R \to S^*$ generalizing the determinant.

Suppose S is a commutative subring of R, and R is a free right S-module of rank m. Left multiplication by an element of R is an S-linear endomorphism of R. This defines the left regular representation $R \to \mathrm{End}_S(R)$ of R as a right S-module, abbreviated l.r.r.(R/S). Henceforth, $\rho: R \to M_m(S)$ will always denote the (faithful) matrix representation of R by l.r.r.(R/S) and a choice of basis. Then $\Delta\rho$ is the determinant of the transfer map: $K_1R \to K_1S$ described in [1, p. 451 (1.7)]; so $\Delta\rho$ will be called a *transfer norm*.

It is useful to know some conditions under which $\Delta\rho$ takes its values in $\mathscr{Z}(R)$. Assume there is a basis B of R over S consisting of commuting units. The conjugation action of B on R extends entrywise and coefficientwise to actions on $M_n(R)$, $R[X]$ and $M_n(R[X])$. Let $(-)^b$ denote $b^{-1}(-)b$ (for b in B) in each case. Assume $S^b \subseteq S$ for every b in B, and let S^B denote the subring of S whose elements commute with all elements of B. Finally, suppose that for any b_1 and b_2 in B, there is a unique b_3 in B for which $b_3^{-1}b_1b_2 \in S$, and that the operation $b_1 \cdot b_2 = b_3$ defines a group structure on B.

Lemma 4. *Under the assumptions in the preceding paragraph, if* $M \in M_n(R)$, *then the characteristic polynomial of* $\rho(M)$ *has coefficients in* S^B, *and* M *is invertible if and only if* $\det \rho(M)$ *is a unit in* S^B.

Proof. If $r \in R$, then $r = \sum_{b \in B} bs_b$ for unique elements s_b in S. If x and $y \in B$, the x, y-entry of $\rho(r)$ is $x^{-1}bys_b^y$, for b with $b \cdot y = x$. If $z \in B$, then $r^z = \sum bs_b^z$, and the x, y-entry of $\rho(r^z)$ is $x^{-1}bys_b^{zy} = (x^{-1}bys_b^y)^z$. So $\rho(r^z) = \rho(r)^z$, and for M in $M_n(R)$, $\rho(M^z) = \rho(M)^z$. Therefore,

$$\begin{aligned}\det(XI_{mn} - \rho(M)) &= \det[\rho(zI_n)(XI_{mn} - \rho(M))\rho(z^{-1}I_n)] \\ &= \det(XI_{mn} - \rho(M^z)) \\ &= \det[(XI_{mn} - \rho(M))^z] \\ &= \det(XI_{mn} - \rho(M))]^z.\end{aligned}$$

This proves the first assertion.

Since $\det \rho(-)$ is $(-1)^{mn}$ times the constant term of the characteristic polynomial of $\rho(-)$, its values lie in S^B. Because it is multiplicative, taking I_n to 1, it takes units to units.

Conversely, suppose $\rho(M) = \mathcal{M}$ has characteristic polynomial $X^{mn} + a_{mn-1} + \cdots + a_o$. If $\det \rho(M)$ is a unit u in S^B, then by the Cayley–Hamilton theorem:

$$\mathcal{M}(\mathcal{M}^{mn-1} + a_{mn-1}\mathcal{M}^{mn-2} + \cdots + a_1 I_{mn})(-1)^{mn+1}u^{-1} = I_{mn}.$$

Note that S^B is central in $M_n(R)$ and $M_{mn}(S)$, and that $\rho : M_n(R) \to M_{mn}(S)$ is an S^B-algebra homomorphism. Thus:

$$\rho(M)\rho[(M^{mn-1} + a_{mn-1}M^{mn-2} + \cdots + a_1 I_n)(-1)^{mn+1}u^{-1}] = \rho(I_n).$$

Since ρ is faithful, M^{-1} has been constructed in $M_n(R)$. $\square$

The motivating example is $R = ZG$ for a finite group G, $S = ZA$ for A a cyclic normal subgroup of G, and $B = T$, a commutative transversal (set of coset representatives) of A in G. In this case the reduced norm and the transfer norm provide two norms from K_1ZG into the center of QG.

4. Reduced norm and transfer norm

Let G be a finite group with cyclic normal subgroup A of order m and quotient $B = G/A$ of order s. Assume there is a commutative transversal T of A in G. In this section we show that $\det \rho : M_n(QG) \to QA$ factors as the reduced norm, ν, followed by a direct sum of norms (each on a summand of $\mathscr{Z}(QG)$).

Let a denote a generator of A. Sending a to $\zeta_{\mathbf{d}} = e^{2\pi i/\mathbf{d}}$, for various $\mathbf{d}$ dividing m, defines a decomposition of rings:

$$QG \cong \bigoplus_{\mathbf{d}|m} Q(\zeta_{\mathbf{d}}) \circ B.$$

The ring $Q(\zeta_{\mathbf{d}}) \circ B$ is a right $Q(\zeta_{\mathbf{d}})$-vector space with basis T. Its multiplication is inherited from QG as follows. Define $\Theta: B \to \mathrm{Aut}(A) \cong (Z/mZ)^*$ by

$$x^{-1}ax = a^{\Theta(b)}, \quad b = xA \in B.$$

Define $\Phi: T \times T \to A$ by

$$t_1t_2 = t_3\Phi(t_1, t_2), \quad t_i \in T.$$

The maps Θ and Φ determine the multiplication in QG. Define $\theta_{\mathbf{d}}$ to be Θ followed by reduction modulo $\mathbf{d}$; define $\phi_{\mathbf{d}}$ to be Φ followed by replacement of a by $\zeta_{\mathbf{d}}$. Then in $Q(\zeta_{\mathbf{d}}) \circ B$, multiplication is determined by the "twist" $\theta_{\mathbf{d}}$:

$$\zeta_{\mathbf{d}}t = t\zeta_{\mathbf{d}}^{\theta_{\mathbf{d}}(b)}, \quad b = tA, \ t \in T,$$

and the "factor set" $\phi_{\mathbf{d}}$:

$$t_1t_2 = t_3\phi_{\mathbf{d}}(t_1, t_2), \quad t_i \in T.$$

Because T commutes, Φ takes its values among elements of A fixed under the conjugation action of B. Denote the kernel of $\theta_{\mathbf{d}}$ by $C_{\mathbf{d}}$ and the image by $L_{\mathbf{d}}$. Regard $L_{\mathbf{d}}$ as a group of automorphisms of $Q(\zeta_{\mathbf{d}})$, and let $F_{\mathbf{d}}$ denote its fixed field. Then $\phi_{\mathbf{d}}$ takes its values in $F_{\mathbf{d}}$.

For now, fix a positive divisor $\mathbf{d}$ of m, and drop $\mathbf{d}$ from the notation. Let H be the union of the cosets in C, and let $Q(\zeta) \circ C$ denote the image of QH under the map replacing a with ζ. Then $Q(\zeta) \circ C$ is the subring of $Q(\zeta) \circ B$ spanned as a $Q(\zeta)$-vector space by the subset of T representing elements of C. The center of $Q(\zeta) \circ B$ is the F-linear span, $F \circ C$, of the same basis. Multiplication: $Q(\zeta) \otimes_F (F \circ C) \to Q(\zeta) \circ C$ is a ring isomorphism, and $F \circ C \subseteq Q(\zeta) \circ C$ is a Galois extension of commutative rings, with Galois group L.

Let T_C be a tranvsersal of C in B, and alter T to obtain a transversal T' of A in G as follows: For cosets in C let T and T' agree. Any other coset, t_0A, in B is uniquely a product of a coset, t_1A, in T_C with a coset, t_2A, in C $(t_i \in T)$. Replace t_0 by t_1t_2. The resulting transversal T' is also commutative, since the image of Φ is central in G. The elements of T' representing the elements of T_C form a basis, V, of $Q(\zeta) \circ B$ over $Q(\zeta) \circ C$ in bijective correspondence with L. And $Q(\zeta) \circ B$ is a crossed product of $Q(\zeta) \circ C$ with L, with a factor set, ϕ', restricting to $V \times V$ that obtained from T'.

The action of L fixes the idempotents which decompose $F \circ C$ as a direct sum, $\bigoplus_i F_i$, of fields; so it respects the corresponding decomposition, $Q(\zeta) \circ C \cong \bigoplus_i E_i$. In fact, L is the Galois group of the Galois extension of commutative rings, $F_i \subseteq E_i$ $(\cong Q(\zeta) \otimes_F F_i)$. The same idempotents decompose $Q(\zeta) \circ B$ as a direct sum of simple components $\bigoplus_i (E_i \circ L)$, where $E_i \circ L$ is the crossed product of E_i with L, with factor set inherited from ϕ'.

With L as a basis, l.r.r.$(Q(\zeta)\circ B/Q(\zeta)\circ C)$ and l.r.r.$(E_i\circ L/E_i)$ yield matrix representations, $\rho_0\colon Q(\zeta)\circ B\to M_l(Q(\zeta)\circ C)$ and $\rho_0'\colon E_i\circ L\to M_l(E_i)$, where l is the order of L (see Section 3). These are applied entrywise in the following commutative diagram, in which the horizontal isomorphisms are induced by entrywise decompositions:

$$\begin{tikzcd}
M_n(Q(\zeta)\circ B) \arrow[r, phantom, "\cong"] \arrow[d, "\rho_0"'] & \bigoplus_i M_n(E_i\circ L) \arrow[d, "\oplus\rho_0'"] \\
M_{nl}(Q(\zeta)\circ C) \arrow[r, phantom, "\cong"] \arrow[d, "\det"'] & \bigoplus_i M_{nl}(E_i) \arrow[d, "\oplus\det"] \\
Q(\zeta)\circ C \arrow[r, phantom, "\cong"] & \bigoplus_i (E_i)
\end{tikzcd}$$

The representation, $\rho_0'\colon E_i\circ L\to M_l(E_i)$, factors as $1\otimes(-)\colon E_i\circ L\to E_i\otimes_{F_i}E_i\circ L$ followed by an isomorphism onto $M_l(E_i)$, as shown by C.T.C. Wall (see [21, Lemma 3.3]). So $\det\rho_0$ is the reduced norm on $M_n(Q(\zeta)\circ B)$.

Suppose we fix an ordering of V and an ordering of the subset, W, of T' representing elements of C, and order the transversal, $T'=VW$, lexicographically. With these ordered bases, if the left regular representations:

$$\rho_0\colon Q(\zeta)\circ B\to M_l(Q(\zeta)\circ C),$$

$$\rho_1\colon Q(\zeta)\circ C\to M_c(Q(\zeta))\quad (c=|C|),$$

$$\rho\colon Q(\zeta)\circ B\to M_{cl}(Q(\zeta))$$

are performed entrywise, then for all $n\geqslant 0$, the following diagram commutes:

$$\begin{tikzcd}
M_n(Q(\zeta)\circ B) \arrow[d, "\rho_0"'] \arrow[dr, "\rho"] & \\
M_{nl}(Q(\zeta)\circ C) \arrow[r, "\rho_1"'] & M_{ncl}(Q(\zeta))
\end{tikzcd}$$

By the transitivity of norms (see [2, p. 140]), the following diagram commutes:

$$\begin{tikzcd}
M_{nl}(Q(\zeta)\circ C) \arrow[rr, "\rho_1"] \arrow[d, "\det"'] & & M_{ncl}(Q(\zeta)) \arrow[d, "\det"] \\
Q(\zeta)\circ C \arrow[r, "\rho_1"'] & M_c(Q(\zeta)) \arrow[r, "\det"'] & Q(\zeta)
\end{tikzcd}$$

Joining these last two diagrams shows that $\det \rho$ is the reduced norm on $M_n(Q(\zeta)\circ B)$ followed by $\det \rho_1 : Q(\zeta)\circ C \to Q(\zeta)$.

With T as a basis, l.r.r. (QG/QA) defines an entrywise map $\rho : M_n(QG) \to M_{ns}(QA)$. Since the choice of T' instead of T (for each $\mathbf{d}$) has no effect on the determinant, $\det \rho : M_n(QG) \to QA$ coincides with:

$$\bigoplus_{\mathbf{d}|m} \det \rho : \bigoplus_{\mathbf{d}|m} M_n(Q(\zeta_{\mathbf{d}})\circ B) \to \bigoplus_{\mathbf{d}|m} Q(\zeta_{\mathbf{d}})$$

So the precise connection between $\det \rho$ and ν is the following:

Theorem 5. *If G is a finite group with a cyclic normal subgroup, A, and a commutative transversal of A in G, then the determinant of entrywise* l.r.r.(QG/QA) *defines a norm,* $\det \rho : M_n(QG) \to QA$, *which is the reduced norm followed by the map N in the commutative diagram:*

$$\begin{array}{ccc}
\mathscr{Z}(QG) \cong \bigoplus_{\mathbf{d}|m} F_{\mathbf{d}}\circ C_{\mathbf{d}} \subseteq & \bigoplus_{\mathbf{d}|m} Q(\zeta_{\mathbf{d}})\circ C_{\mathbf{d}} \\
\Big\downarrow N & \Big\downarrow \oplus n_{\mathbf{d}} \\
QA \quad \cong & \bigoplus_{\mathbf{d}|m} Q(\zeta_{\mathbf{d}})
\end{array}$$

where $n_{\mathbf{d}}$ is determinant of l.r.r.$(Q(\zeta_{\mathbf{d}})\circ C_{\mathbf{d}}/Q(\zeta_{\mathbf{d}}))$.

5. Results for metacyclic groups

There are two advantages to the use of $\det \rho$ instead of ν. One is that $\det \rho$ is easier to compute. The other is that $\nu M_n(ZG)$ is unknown, except that it lies in $\nu M_n(QG)$ and in the maximal Z-order in $\mathscr{Z}(QG)$; but $\det \rho M_n(ZG)$ is more easily determined. By Lemma 4, $\det \rho M_n(ZG)$ is contained in $ZA \cap \mathscr{Z}(QG) = (ZA)^B$, the fixed points in ZA under the conjugation action of B $(=G/A)$. For each x in A, let $\operatorname{orb}(x)$ denote the sum of the distinct elements in the orbit of x under B-action. Then $(ZA)^B$ is a free Z-module based on $\{\operatorname{orb}(x), x \in A\}$. Let D denote the fixed points of A under B-action; these are the x in A with $\operatorname{orb}(x) = x$. For each y in ZA let $\operatorname{trace}(y) = \sum_{z\in B} y^z$. When B acts faithfully on the orbit of x, $\operatorname{orb}(x) = \operatorname{trace}(x)$. Note that trace is a ZD-module endomorphism of ZA.

When the action of B is either trivial or faithful on each orbit in A, then $(ZA)^B = ZD + \operatorname{trace} ZA$. When the canonical map $G \to G/A$ $(=B)$ splits, then $1 + \operatorname{trace} ZA \subseteq \det \rho ZG$. (This is seen by using B as the basis of ZG over ZA, and using row and column reduction to prove $\det \rho(1 + \sum_{z\in B} zy) = 1 + \operatorname{trace}(y)$ for any y in ZA.) When D is a direct factor of A, the projection: $A \to D$ induces a ring endomorphism, δ, of ZG, with useful properties. These considerations motivate the hypotheses in the following theorem:

Theorem 6. *Let G denote a metacyclic group with presentation $\langle a, b\colon a^m, b^s, b^{-1}aba^{-\alpha}\rangle$, where m divides $\alpha^s - 1$. If $d = (m, \alpha - 1)$ and $e = m/d$, assume $(d, e) = 1$ and the multiplicative order of α modulo p is s for every prime p dividing e. Assume $SK_1ZG = 1$. Let A, B and D be the cyclic subgroups generated by a, b and a^e respectively. Then $K_1ZG = s_1(\mathcal{V})$, where $\mathcal{V}$ is the subgroup of ZG^* generated by $Z[BD]^*$ and units in $1 + (\sum_{z\in B} z)ZA$.*

Proof. The center of G is the cyclic normal subgroup D of order d generated by a^e. Let E denote the subgroup of order e generated by a^d. Then $A = E \times D$, and $G = (BE) \times D$.

Both A and D are cyclic normal subgroups which can play the role of A in Theorem 5. The corresponding decompositions of QG are connected. The decomposition

$$\lambda : QG \cong \bigoplus_{t|m} Q(\zeta_t) \circ B \quad (a \to (\zeta_t))$$

factors as the decomposition:

$$\mu : QG \cong \bigoplus_{v|e} Q(\zeta_v) \circ BD \quad (a^d \to (\zeta_v))$$

followed by the decompositions:

$$\eta_v : Q(\zeta_v) \circ BD \cong \bigoplus_{w|d} Q(\zeta_{vw}) \circ B,$$

where $\zeta_v \mapsto \zeta_{vw}^d$, $b \mapsto b$ and $a^e \mapsto \zeta_{vw}^e$.

There is a commutative diagram:

$$\begin{tikzcd}
M_n(QG) \arrow[r, "\mu"] \arrow[dd, "\nu"'] & \bigoplus\limits_{v} M_n(Q(\zeta_v) \circ BD) \arrow[r, "\eta"] \arrow[d, "\oplus \rho_0"] & \bigoplus\limits_{v,w} M_n(Q(\zeta_{vw}) \circ B) \arrow[d, "\oplus \rho_0"] \\
 & M_n(QBD) \oplus \bigoplus\limits_{v>1} M_{ns}(Q(\zeta_v)D) \arrow[r] \arrow[d, "\oplus \det"] & \bigoplus\limits_{w} M_n(Q(\zeta_w)B) \oplus \bigoplus\limits_{\substack{v>1\\ w}} M_{ns}(Q(\zeta_{vw})) \arrow[d, "\oplus \det"] \\
Z(QG) \arrow[r] \arrow[d, "N"'] & QBD \oplus \bigoplus\limits_{v>1} Q(\zeta_v)D \arrow[r] \arrow[d, "\oplus N_v"] & \bigoplus\limits_{w} Q(\zeta_w)B \oplus \bigoplus\limits_{\substack{v\geq 1\\ w}} Q(\zeta_{vw}) \arrow[d, "\oplus n_{vw}"] \\
QA \arrow[r] & QD \oplus \bigoplus\limits_{v>1} Q(\zeta_v))D \arrow[r] & \bigoplus\limits_{w} Q(\zeta_w) \oplus \bigoplus\limits_{\substack{v>1\\ w}} Q(\zeta_{vw})
\end{tikzcd}$$

where the indices **v** and **w** in each direct sum range over the divisors of e and d respectively. The map η is $\eta_{\mathbf{v}}$ entrywise in each **v** coordinate; so $\eta\mu = \lambda$. The maps, ρ_0, are the representations, ρ_0, in the proof of Theorem 5, where the cyclic normal subgroup, A, is taken to be D in the second column of the diagram and A in the third. The maps, $n_{\mathbf{vw}}$, are identity maps for $\mathbf{v}>1$ and norms for $\mathbf{v}=1$. The maps, $N_{\mathbf{v}}$, are defined to correspond to $\bigoplus_{\mathbf{w}} n_{\mathbf{vw}}$; so $N_{\mathbf{v}}$ is the identity map for $\mathbf{v}>1$, and N_1 is $\det \rho$, where ρ is l.r.r.(QG/QA) with basis B.

Suppose M is an element of the kernel of $\det \rho: \mathrm{GL}_n QC \to QA^*$. Then $\mu\nu(M) = (u, 1, \ldots, 1)$, where $u \in QBD^*$ and $\det \rho(u) = 1$. The composite of the inclusion: $QBD \subseteq QG$ and μ is the diagonal map: $QBD \to \bigoplus_{\mathbf{v}} Q(\zeta_{\mathbf{v}}) \circ BD$. For $\mathbf{v}=1$, $\det \rho_0$ is the identity map; for $\mathbf{v}>1$, $\det \rho_0$ agrees with $\det \rho$ on QBD. Therefore $\mu\nu(u) = (u, 1, \ldots, 1) = \mu\nu(M)$. Since μ is injective, $\nu(u) = \nu(M)$.

Note that u is obtained from M by entrywise application of the Q-algebra map $\delta: QG \to QG$ defined by $\mathrm{a}^{\mathrm{d}} \mapsto 1$, $\mathrm{a}^{\mathrm{e}} \mapsto \mathrm{a}^{\mathrm{e}}$ and $b \mapsto b$, followed by the determinant. So if M is in $\mathrm{GL}_n ZG$, u is in ZG^*. By hypothesis, $SK_1 ZG = 1$; so u and M represent the same element of $K_1 ZG$. Thus the kernel of $\Delta\rho: K_1 ZG \to (ZA)^{B*}$ is represented by units from ZBD^*.

Now consider the image of $\Delta\rho$. Since $K_1 ZG = s_2 \mathrm{GL}_2 ZG$, we also have $\Delta\rho K_1 ZG = \det \rho \mathrm{GL}_2 ZG$. If entrywise application of δ is also called δ, then δ and ρ commute. So

$$\begin{aligned}\delta \det \rho \mathrm{GL}_2 ZG &= \det \delta\rho \mathrm{GL}_2 ZG = \det \rho \mathrm{GL}_2 ZBD \\ &= \det \rho \det \mathrm{GL}_2 ZBD = \det \rho ZBD^*,\end{aligned}$$

where the third step is transitivity of norms, using the commutativity of ZBD (see [2, p. 140]).

Suppose $q + \mathrm{trace}(p) \in \Delta\rho K_1 ZG$, where $q \in ZD$ and $p \in ZA$. Then

$$q + s\delta(p) = \delta(q + \mathrm{trace}(p)) = u$$

for some u in $\det \rho ZBD^*$. So

$$q + \mathrm{trace}(p) = u + \mathrm{trace}(p - \delta(p)) = u(1 + \mathrm{trace}(u^{-1}[p - \delta(p)])),$$

which is an element of $\det \rho(\mathcal{V})$.

If $x \in K_1 ZG$, then $\Delta\rho(x) \in \det \rho(V) = \Delta\rho s_1(V)$. Since the kernel of $\Delta\rho$ is in $s_1(\mathcal{V})$, so is x. $\square$

Note 7. The hypotheses of Theorem 6 are satisfied (including $SK_1 ZG = 1$) for metacyclic groups: $\langle a, b\colon a^m, b^s, b^{-1}aba^{-\alpha}\rangle$ (where m divides $\alpha^s - 1$) of the following two types:

(a) The integer s is a prime not dividing e.

(b) The integers s and e are powers of distinct primes q and p respectively, q^2 does not divide d, and the multiplicative order of α modulo p is s.

To see that $(d, e) = 1$ in these groups, note that since m divides $\alpha^s - 1$, any prime dividing both d and e divides both $\alpha - 1$ and $1 + \alpha + \cdots + \alpha^{s-1}$; so it divides s. For proofs that $SK_1 ZG = 1$ for these groups, see [16].

Example 8. Let D_5 denote the dihedral group with presentation $\langle a, b: a^5, b^2, b^{-1}aba\rangle$. Then K_1ZD_5 is represented by the units $ZBD^* = ZB^* = \{\pm 1, \pm b\}$ times units of the form: $1+(1+b)y$ for y in ZA.

Note 9. If we let $\mathscr{Y}$ denote the product of the ZH^* as H ranges over all proper subgroups of D_5, then $\mathscr{Y}$ is generated by a, b, -1 and an element $u = 1-a-a^{-1}$ of infinite order (see [5, p. 116]). So $\det \rho(\mathscr{Y})$ is generated by ± 1 and u^2, and does not contain u. However, $u = \det \rho(1-a-ba) \in \det \rho(\mathscr{V})$. Since $\det \rho: ZG^* \to (ZA)^{B*}$ factors as $\Delta\rho s_1$, it follows that $s_1(\mathscr{Y}) \subsetneq s_1(\mathscr{V}) = K_1ZD_5$. We conclude from this example that $K_1Z[-]$ *does not satisfy induction on elementary subgroups*. That is, if $\mathscr{E}$ is the set of all subgroups of G of the form $C \times P$, where C is cyclic and P has prime power order, then in the notation of Theorem 3 and its proof, $\prod_{E\in\mathscr{E}} i.\ K_1ZE$ generally is a proper subset of K_1ZG.

6. Dihedral groups

Theorem 6 applies to some, but not all dihedral groups $D_m \cong \langle a, b: a^m, b^2, b^{-1}aba\rangle$. An obstacle to the proof arises when m is divisible by 4, so that D is not a direct factor of A. But since s is only 2, an alternative argument yields similar results:

Theorem 10. *Let $D_m \cong \langle a, b: a^m, b^2, b^{-1}aba\rangle$, and let A denote the subgroup generated by a. Then $K_1ZD_m = s_1(\mathscr{W})$, where $\mathscr{W}$ is the subgroup of ZD_m^* generated by -1, a, b and units in $1+(1+b)ZA$.*

Proof. By Theorem 5, $\Delta\rho = (\bigoplus_{\mathbf{d}|m} n_{\mathbf{d}})\nu$. Now νK_1ZD_m lies in the group of units in the integral closure of Z in $\bigoplus_{\mathbf{d}|m}Q(\zeta_{\mathbf{d}})C_{\mathbf{d}}$, the domain of $\bigoplus n_{\mathbf{d}}$. For $\mathbf{d}>2$, $n_{\mathbf{d}}$ is the identity map on $Q(\zeta_{\mathbf{d}})$. If $\mathbf{d}=1$ or 2, $Q(\zeta_{\mathbf{d}})C_{\mathbf{d}} = QB \cong Q\times Q$; so these components of νK_1ZD_m are in $(Z\times Z)^*$, a finite group. Thus νK_1ZD_m intersects the kernel of $\bigoplus n_{\mathbf{d}}$ in a finite group. Since $SK_1ZD_m = 1$ for any dihedral group (see [15], [11] and [17]), ν is injective on K_1ZD_m. So the kernel of $\Delta\rho: K_1ZD_m \to (ZA)^{B*}$ is finite. By a deep result of C.T.C. Wall (see [21, Proposition 6.5]), for finite groups, G, the torsion part of K_1ZG is $s_1(\pm G)\times SK_1ZG$. So the kernel of $\Delta\rho$ on K_1ZD_m is represented by trivial units.

The image, $\Delta\rho K_1ZD_m$, is contained in the units of $(ZA)^B$, which is $Z+\text{trace } ZA$ when m is odd, and $Z+ZA^{m/2}+\text{trace } ZA$ when m is even. For any M in $M_2(ZD_m)$, $\rho(M)$ is a 4×4 matrix in 2×2 blocks of the form:

$$\begin{bmatrix} p & q^b \\ q & p^b \end{bmatrix} \qquad (p, q \text{ in } ZA).$$

Inspection of $\det \rho(M)$ shows it to be a sum of terms of the forms $\pm rr^b$ and $r+r^b$ (r in ZA). When m is even, the coefficient of $a^{m/2}$ in such terms is even. So $\Delta\rho K_1ZD_m$ lies in $Z+\text{trace } ZA$ for any m, even or odd. The augmentation map: $ZA \to Z$ takes units to ± 1 and takes trace ZA into $2Z$. So $\Delta\rho K_1 ZD_m$ lies in $1+2Z+\text{trace } ZA = 1+\text{trace } ZA = \det\rho(1+(1+b)ZA)$. □

Note 11. If $y \in ZA$, $\det\rho$ has the same effect on $1+(1-b)y$ and $1+(1+b)y$. So in Theorem 19, we may replace $\mathscr{W}$ by the group generated by trivial units and units in $1+(1-b)ZA$. This change is technically useful in the next section.

7. Application to CW-complexes

In the following discussion all CW-complexes are pointed, all maps preserve base point, and all homotopies are relative to base points. For any group G, a *G-complex* is a connected finite two-dimensional CW-complex with fundamental group G. The following classification of homotopy types of G-complexes is found in [6] and in [8]. The graph whose vertices are homotopy types of G-complexes and whose edges connect the homotopy type of X to the homotopy type of $X \vee S^2$ is an infinite tree, $\mathrm{HT}(G,2)$. The Euler characteristic makes this graph a preorder. If $\chi_{\min} = \min\{\chi(x), X \text{ a } G\text{-complex}\}$, the *level* of a G-complex Y is $\chi(Y)-\chi_{\min}$.

If G is a finite group, there is (by [8] and [3]) exactly one homotopy type at each level greater than zero. The number of homotopy types at level zero is not generally known. When G is cyclic, or dihedral of order $2m$ (m odd), then there is exactly one homotopy type at level zero as well (see [8, p. 223]).

The classification of simple homotopy types within homotopy types of G-complexes is aided by the following results. If X is a finite connected CW-complex, let $\mathscr{E}(X)$ denote the group of homotopy classes of homotopy self-equivalences of X. Whitehead torsion defines a map, $\tau:\mathscr{E}(X)\to \mathrm{Wh}_1\pi_1(X)$, where $\mathrm{Wh}_1 G = K_1ZG/s_1(\pm G)$. The following result is due to W.H. Cockroft and R.M.F. Moss (see [4, p. 80]):

Theorem A. *Every* CW-*complex homotopy equivalent to X is simple homotopy equivalent to X if and only if $\tau:\mathscr{E}(X)\to \mathrm{Wh}_1\pi_1(X)$ is surjective.*

Let iS^n denote the bouquet of i copies of the n-sphere S^n. M.N. Dyer has established the next result (see [9, Theorem 2]):

Theorem B. *If X is a finite connected* CW-*complex with fundamental group G, and if stabilization*: $\mathrm{GL}_i ZG \to K_1 ZG$ *is surjective, then $\tau:\mathscr{E}(X\vee iS^n)\to \mathrm{Wh}_1 G$ is surjective for all $n \geqslant 2$.*

Consequently, if G is finite, there is exactly one simple homotopy type of G-complexes at each level greater than one. And when $K_1ZG = s_1(ZG^*)$, there is exactly one simple homotopy type at level one as well.

If G has a finite presentation, $\mathcal{P} = \langle g_1, \ldots, g_k : r_1, \ldots, r_n \rangle$, where each r_i is a reduced word, the *cellular model* P of $\mathcal{P}$ is the G-complex with a single 0-cell, an oriented 1-cell for each generator, g_i, and a 2-cell for each relator, r_j, attached to the 1-skeleton following the letters of r_j. Each simple homotopy type of G-complexes contains the cellular model of some presentation of G (by [6, Proposition 1]). The Euler characteristic, $\chi(P)$, is $1-\delta$ where $\delta = k-n$ is the "deficiency" of the presentation $\mathcal{P}$. For a finite group G the largest possible deficiency is zero; so $\chi_{\min} \geqslant 1$. If the deficiency of $\mathcal{P}$ is maximum for presentations of G, then P has level zero in $\mathrm{HT}(G, 2)$.

The following theorem is a slightly more general form of a result of M.N. Dyer (see [9, Theorem 4]). The proof is essentially the same as his argument.

Theorem C. *Let G be a group with presentation $P = \langle a, b \colon a^m, b^s, b^{-1}aba^{-\alpha} \rangle$. If P is the cellular model of $\mathcal{P}$, then the image of $\tau \colon \mathscr{E}(P) \to \mathrm{Wh}_1 G$ contains all elements represented by units of ZG.*

Proof. The cellular chain complex, $C_*(\tilde{P})$, of the universal cover, $\tilde{P}$, of P is:

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

where C_2, C_1, and C_0 are free ZG-modules of ranks 3, 2 and 1 respectively. With a suitable choice of bases, ∂_2 is represented by the matrix of Fox derivatives obtained from $\mathcal{P}$ (see [10, p. 198]):

$$\Sigma = \begin{bmatrix} 1 + a + \cdots + a^{m-1} & 0 & * \\ 0 & 1 + b + \cdots + b^{s-1} & * \end{bmatrix}$$

where the entries $*$ lie in ZG.

Let $\varepsilon \colon ZG \to Z$ be the augmentation map, and denote by $\sigma \colon ZG \to ZB$ the partial augmentation sending a to 1. Suppose u is in ZG^*, and $v = u\varepsilon(u)$. Denote the corresponding elements of $\mathrm{Wh}_1 G$ by $[u]$ and $[v]$. If Γ is the matrix:

$$\Gamma = \begin{bmatrix} v\sigma(v)^{-1} & 0 & 0 \\ 0 & \sigma(v) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $\Sigma\Gamma = \Sigma$. So there is a chain isomorphism $g \colon C_*(\tilde{P}) \to C_*(\tilde{P})$ given by the identity map on C_i $(i \neq 2)$ and multiplication by Γ on C_2. By [14, Theorem 3], there is a map f in $\mathscr{E}(P)$ with $\tilde{f}$ (in $\mathscr{E}(\tilde{P})$) inducing g. Then $\tau(f) = \tau(g) = [v] = [u]$, since the matrix

$$\begin{bmatrix} \sigma(v)^{-1} & 0 & 0 \\ 0 & \sigma(v) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is elementary, and $\varepsilon(u)$ is a trivial unit. $\square$

When $G = D_m$ (m odd) a complete classification of simple homotopy types of G-complexes is possible. There is a deficiency zero presentation of D_m given by

$$\mathscr{R} = \langle a, b : a^m b^{-2}, [b, a^{(m-1)/2}]a^{-1} \rangle$$

so its cellular model R has level zero. By Theorems A and B, and since $K_1 ZD_m = s_1(ZD_m^*)$, simple homotopy type and homotopy type coincide at each level above zero. We next show they also coincide at level zero:

Theorem 12. *If D_m is the dihedral group of order $2m$ and m is odd, then two D_m-complexes are simply homotopy equivalent if and only if they have the same Euler characteristic.*

Proof. By the above discussion it is sufficient to prove that $\tau : \mathscr{E}(R) \to \mathrm{Wh}_1 D_m$ is surjective. The cellular chain complex $C_*(\tilde{R})$ is:

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

where C_2, C_1 and C_0 are free ZD_m-modules of ranks 2, 2 and 1 respectively. With proper choice of bases, ∂_2 is represented by the matrix of Fox derivatives:

$$\Sigma = \begin{bmatrix} 1 + a + \cdots + a^{m-1} & * \\ -(1+b) & * \end{bmatrix}$$

where the entries $*$ are in ZD_m.

The elements of $\mathrm{Wh}_1 D_m$ are all represented by units in ZD_m^* of the form $u = 1 + (1-b)p(a)$ where $p(X) \in Z[X]$ (see Note 11, Section 6). Under the partial augmentation, $\sigma : ZD_m \to ZB$ ($B = \{1, b\}$), taking a to 1, such a unit, u, is sent into $ZB^* = \{\pm 1, \pm b\}$. Let $v = u\sigma(u)$. Then

$$(1 + a + \cdots + a^{m-1})v = (1 + a + \cdots + a^{m-1})\sigma(u)^2 = 1 + a + \cdots + a^{m-1},$$

and

$$(1+b)v = (1+b)\sigma(u) = (1+b)\varepsilon(u) = 1 + b.$$

So if Γ is the matrix:

$$\Gamma = \begin{bmatrix} v & 0 \\ 0 & 1 \end{bmatrix}$$

then $\Sigma\Gamma = \Sigma$. The rest of the proof is similar to the end of the proof of Theorem C. $\square$

Note 13. In [8] the tree $\mathrm{HT}(G, i)$ is defined for (G, i)-complexes. It is known that D_m (m odd) has minimal free period 4, and that $\mathrm{HT}(D_m, 4n-2)$ is a single stalk for $n > 0$ (see [7, 10.1]). By stacking resolutions one can use the argument proving Theorem

12 to show that two $(D_m, 4n-2)$-complexes have the same simple homotopy type if and only if they have the same Euler characteristic.

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