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# ON 2-DIMENSIONAL $CW$-COMPLEXES WITH A SINGLE 2-CELL

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**In this paper we are interested in finite connected 2-dimensional $CW$-complexes, each with a single 2-cell. We show any two such complexes have the same homotopy type if their fundamental groups are isomorphic. In fact, there is a homotopy equivalence inducing any isomorphism of the fundamental groups. We also study the homotopy factorizations of such spaces into finite sums.**

In this paper we are interested in finite connected 2-dimensional $CW$-complexes with a single 2-cell. Each such $CW$-complex has the homotopy type of the cellular model $C(\mathscr{R})$ of some finite one-relator presentation

$$\mathscr{R} = (x_1, \cdots, x_n : R)$$

of $\Xi = \pi_1 X$. If the single relator $R$ is not a proper power, it is known that the cellular model $C(\mathscr{R})$ is aspherical (see [10], [1], or [4]), hence it is determined up to homotopy type by its fundamental group. If the single relator $R$ is a proper power, $C(\mathscr{R})$ is not aspherical, nevertheless we are able to prove the following:

THEOREM 1. *Any two finite connected 2-dimensional $CW$-complexes, each with a single 2-cell, have the same homotopy type if their fundamental groups are isomorphic. In fact there is a homotopy equivalence inducing any isomorphism of the fundamental groups.*

Our proof makes use of Lyndon's resolution for one-relator groups [10] and some combinatorial results on one-relator groups which can be found in the book by Magnus, Karass, and Solitar [11].

Theorem 1 has these corollaries:

COROLLARY 1. *Let $X$ and $Y$ be two finite connected 2-dimensional $CW$-complexes, each with a single 2-cell. Then $X \simeq Y$ if $X \vee L \simeq Y \vee M$ where $L$ and $M$ are finite $CW$-complexes with isomorphic fundamental groups. Thus $X \simeq Y$ if and only if $X \vee L \simeq Y \vee L$ where $L$ is any finite $CW$-complex.*

*Proof.* We have $\pi_1 X * \pi_1 L \approx \pi_1 Y * \pi_1 M$. Because all groups involved are finite generated, we can write these as free product of irreducible groups (relative to free product), and by uniqueness of such free product decompositions (see [11], p. 245), we obtain $\pi_1 X \approx \pi_1 Y$. The result now follows from Theorem 1.

Given a space $X$ with fundamental group $\Xi$, the homotopy classes of homotopy self-equivalences $X \to X$ form a group under composition. There is an evaluation homomorphism

$$\sharp\colon \mathscr{E}(X) \to \operatorname{Aut} \Xi$$

which assigns to each based self-equivalence $f\colon X \to X$ the automorphism $f_\sharp\colon \pi_1 X = \Xi \to \Xi$ in $\operatorname{Aut} \Xi$. By Theorem 1 we have

COROLLARY 2. *For a finite connected 2-dimensional CW-complex $X$ with a single 2-cell, the evaluation homomorphism $\sharp\colon \mathscr{E}(X) \to \operatorname{Aut} \Xi$ is an epimorphism with kernel $H^2(\Xi, \pi_2 X)$. (See Schellenberg [12].)*

The only possible free product decompositions $\Xi \approx H * K$ of a finitely generated one-relator group $\Xi$ involve another such group $H$ and a free group $K$ of finite rank (this statement follows from a remark in [13] (page 276) which is stated there without proof, hence we include its proof in the proof of Theorem 2). We prove the following topological analogue of this algebraic situation:

THEOREM 2. *The only possible nontrivial homotopy decompositions $X \simeq W \vee Z$ of a connected finite 2-dimentional CW-complex with a single 2-cell involves another such complex $W$ and a finite sum $Z = kS^1$ of $k$ copies of the 1-sphere $S^1$, and there is such a homotopy decomposition $X \simeq W \vee Z$ for each nontrivial free product decomposition $\pi_1 X \approx H * K$.*

DEFINITION. We say a space $X$ is *irreducible* if each homotopy decomposition $X \simeq Y \vee Z$ is trivial, i.e., either $Y$ or $Z$ is contractible.

By Theorem 2 we have that a finite connected 2-dimensional CW-complex $X$ with a single 2-cell is irreducible if and only if $\pi_1 X$ is irreducible (see also Lemma 3 in §3). In [13] Shenitzer proves some results which ensure the irreducibility of a one-relator group. For example he shows that the one-relator group

$$\left(x_1, \cdots, x_k\colon \left(\prod_{i=1}^{k} x_i^2\right)^q\right)$$

is irreducible, hence its cellular model is irreducible. In particular any nonorientable closed surface of genus $k \geqq 1$ is irreducible.

For a reducible one-relator group $\Xi$, by uniqueness of the free product decompositions, we have that $\Xi$ can be written as a free product $H*K$ where $H$ is an irreducible one-relator group and $K$ is a free group of rank $k$, for some maximal integer $k \geqq 1$. We have the following topological analogue.

COROLLARY 3. *If $X$ is a finite connected 2-dimentional CW-complex with a single 2-cell, then $X \simeq Y \vee kS^1$ where $Y$ is an irreducible 2-dimensional CW-complex with a single 2-cell and $k \geqq 0$ is the maximal number of free factors in a free product decomposition of $\pi_1 X$.*

We have the following uniqueness result for the decompositions relative to the sum:

COROLLARY 4. *Suppose $X_1 \vee X_2 \vee \cdots \vee X_n \simeq Y_1 \vee Y_2 \vee \cdots \vee Y_m$ where $X_i$ and $Y_j$ are 2-dimensional finite connected irreducible CW-complexes with a single 2-cell. Then $n = m$ and $Y_1, \cdots, Y_n$ can be rearranged so as to yield $Y_{j_1}, \cdots, Y_{j_n}$ where $X_i \simeq Y_{j_i}$.*

*Proof.* We have $\pi_1 X_1 * \pi_1 X_2 * \cdots * \pi_1 X_n \cong \pi_1 Y_1 * \pi_1 Y_2 * \cdots * \pi_1 Y_m$ where $\pi_1 X_i$ and $\pi_1 Y_j$ are irreducible with respect to free product. Thus by uniqueness of such free product decompositions, we have $n = m$ *and* $\pi_1 X_i \approx \pi_1(Y_{j_i})$. The result now follows from Theorem 1.

The organization of this paper is as follows. The proof of Theorem 1 is given in §2, using two lemmas which are given in §1. The proof of Theorem 2 is given in §3. Finally in §4 we give an example of Dunwoody which shows that the Theorem 1 fails to generalize for 2-dimensional $CW$-complexes with one-relator fundamental groups and the same number $n > 1$ of 2-cells.

All the spaces in this paper are connected $CW$-complexes unless otherwise stated, with some zero cell chosen as basepoint which is preserved by all maps and homotopies.

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**1. Some results about one-relator groups.** A finite presentation $\mathscr{P} = (g_\alpha : r_\beta)$ consists of a finite set $\{g_\alpha\}$ of elements, called the generators of $\mathscr{P}$, together with a finite set $\{r_\beta\}$ of elements in the free group $F = F(g_\alpha)$ on the generators, called the relators of $\mathscr{P}$.

The group presented by $\mathscr{P} = (g_\alpha : r_\beta)$ is the quotient group $\pi = F/N$ of $F$ modulo the smallest normal subgroup $N = N(r_\beta)$ of $F$ containing the relators $r_\beta$. In this case we say $\pi$ is a finitely presented group.

Now we record some results about the one-relator group $\Xi$ which is given by the presentation

$$\mathscr{R} = (x_1, \cdots, x_n : R^r)$$

where $R$ is not a proper power.

*Notation.* For simplicity, we employ the same notation for elements of $F$ and $\Xi$. We let $Z\Xi$ denote the integral group ring of $\Xi$. All $Z\Xi$-modules are left $Z\Xi$-modules. Any element $w \in Z\Xi$ defines a left $Z\Xi$-module homomorphism $w: Z\Xi \to Z\Xi$ given by the right multiplication. If $K$ is any left $\Xi$-module and $w \in Z\Xi$, ${}_wK$ denotes the subgroup of all $k \in K$ such that $wk = 0$. For $w \in \Xi$ and a positive integer $s$, we let

$$\langle w, s\rangle = 1 + w + \cdots + w^{s-1} \quad \text{and} \quad \langle w, -s\rangle = -w^{-s}\langle w, s\rangle \quad \text{in} \quad Z\Xi .$$

We have the following $\langle \ \ \rangle$-identities:

$$(w-1)\langle w, s\rangle = w^s - 1 , \quad \langle w, s\rangle + w^s\langle w, t\rangle = \langle w, s+t\rangle ,$$
$$\langle w, s\rangle\langle w^s, t\rangle = \langle w, st\rangle$$

whenever the elements involved are defined. (See [12].)

The following is a $\Xi$-resolution of the trivial $\Xi$-module $Z$ (see Lyndon [10]):

$$\cdots \xrightarrow{\langle R, r\rangle} Z\Xi \xrightarrow{R-1} Z\Xi \xrightarrow{\langle R, r\rangle} Z\Xi \xrightarrow{R-1} Z\Xi$$
$$\xrightarrow{\partial_2} (Z\Xi)^n \xrightarrow{\partial_1} Z\Xi \xrightarrow{\varepsilon} Z \longrightarrow 0$$

where $\varepsilon: Z\Xi \to Z$ is the augmentation homomorphism,

$$\partial_1 = (x_1 - 1, \cdots, x_n - 1) \quad \text{and} \quad \partial_2 = \langle R, r\rangle(\partial R/\partial x_1, \cdots, \partial R/\partial x_n)$$

is the Jacobian matrix of the presentation $\mathscr{R}$ described in the free differential calculus of R. H. Fox [5, p. 198].

Hence using the left ideal $Z\Xi(R-1)$ as the coefficient module and the above resolution, there is the cohomology group

$$H^3(\Xi, Z\Xi(R-1)) = {}_{\langle R,r\rangle}Z\Xi(R-1)/(R-1)Z\Xi(R-1) .$$

LEMMA 1. *The cohomology group*

$$H^3(\Xi, Z\Xi(R-1)) \approx Z\rho(R-1)/Z\rho(R-1)^2 \approx Z_r$$

*where $\rho$ denotes the cyclic subgroup of $\Xi$ generated by $R$.*

*Proof.* Let $w \in Z\Xi$. Then

$$\begin{aligned}\langle R, r\rangle w(R-1) = 0 &\Longleftrightarrow w(R-1) \in (R-1)Z\Xi \\ &\qquad \text{[from Lyndon's resolution]} \\ &\Longleftrightarrow w \in Z\rho + Z\Xi\langle R, r\rangle + (R-1)Z\Xi \\ &\qquad \text{[This is Lemma 3 of Hughes [8]]} .\end{aligned}$$

Thus

$$\begin{aligned}H^3(\Xi, Z\Xi(R-1)) &= (Z\rho(R-1) + (R-1)Z\Xi(R-1))/(R-1)Z\Xi(R-1) \\ &= Z\rho(R-1)/Z\rho(R-1)^2 .\end{aligned}$$

Now the second isomorphism of the lemma follows from the following relation: $R^i(R-1) \equiv (R-1)$ modulo $(R-1)^2$. The proof is via induction. For $i = 0$, the result is trivial and for $i = 1$, the relation is simply $R^2 - R \equiv R - 1$ modulo $R^2 - 2R + 1$. Suppose it is true for $i = n - 1 \geqq 1$, then $R^n(R-1) = R \cdot R^{n-1}(R-1) \equiv R(R-1) \equiv (R-1)$ modulo $(R-1)^2$. One can therefore define the required isomorphism this way:

$$\Sigma \alpha_i R^i(R-1) \bmod Z\rho(R-1)^2 \longrightarrow \Sigma \alpha_i \bmod r .$$

That $H^3(\Xi, Z\Xi(R-1)) \approx Z_r$ also follows from Theorem 2, page 129 of [6].

LEMMA 2. *Let $(r, s) = 1$. Then*

( i ) *The left ideals $Z\Xi(R-1)$ and $Z\Xi(R^s-1)$ in $Z\Xi$ coincide.*

( ii ) *The $Z\Xi$-module homomorphism $\langle R, s\rangle$: $Z\Xi(R-1) \rightarrow Z\Xi(R^s-1)$ is an isomorphism and the induced homomorphism $\langle R, s\rangle_*$: $Z_r \approx H^3(\Xi, Z\Xi(R-1)) \rightarrow H^3(\Xi, Z\Xi(R-1)) \approx Z_r$ carries $1 \rightarrow s$.*

*Proof.* ( i ) Because $(r, s) = 1$, there exists positive integers $k$ and $s'$ such that $ss' = 1 + kr$. Using the $\langle\ \ \rangle$-identities, we obtain

$$\begin{aligned}\langle R^s, s'\rangle(R^s - 1) &= \langle R^s, s'\rangle\langle R, s\rangle(R-1) \\ &= \langle R, ss'\rangle(R-1) \\ &= (k\langle R, r\rangle + 1)(R-1) \\ &= R - 1 ,\end{aligned}$$

hence $Z\Xi(R-1)$ is a subset of $Z\Xi(R^s-1)$. Since $\langle R, s\rangle(R-1) = R^s - 1$, we have $Z\Xi(R^s-1)$ is a subset of $Z\Xi(R-1)$.

( ii ) One easily checks that when $ss' \equiv 1 \bmod r$, the $Z\Xi$-module homomorphisms $\langle R, s\rangle$ and $\langle R^s, s'\rangle$ are inverses. In terms of the identifications of Lemma 1, the induced cohomology homomorphism

$\langle R, s\rangle_*$ is given by

$$1(R-1) \bmod Z\rho(R-1)^2 \longrightarrow \langle R, s\rangle(R-1) \bmod Z\rho(R-1)^2$$

or equivalently,

$$1 \bmod r \longrightarrow s \bmod r .$$

2. **Proof of Theorem 1.** Given a 2-dimensional $CW$-complex $X$ with a single 0-cell, the universal covering $\tilde{X}$ of $X$ admits the fundamental group $\Xi = \pi_1 X$ as the group of covering transformations, and there is a canonical $CW$-structure on $\tilde{X}$ for which the projection map is cellular and the covering transformations $g: \tilde{X} \to \tilde{X}$, $g \in \Xi$, are orientation preserving cellular homeomorphisms. The action of the covering transformations on the cellular chain complex $C_*(\tilde{X})$ via the induced chain maps $g_*: C_*(\tilde{X}) \to C_*(\tilde{X})$, makes $C_*(\tilde{X})$ a chain complex over $Z\Xi$. We can identify the second homotopy module $\pi_2 X$ with $H_2\tilde{X} = \ker \partial_2(\tilde{X})$, using the covering projection isomorphism $\pi_2\tilde{X} \approx \pi_2 X$ and the Hurewicz isomorphism $\pi_2\tilde{X} \approx H_2\tilde{X}$.

Now let $Y$ be any other 2-dimensional $CW$-complex with a single 0-cell, and let $\alpha$ be homomorphism from $\pi_1 X = \Xi \to \pi = \pi_1 Y$. Let ${}_\alpha C_*(\tilde{Y})$ denote $C_*(Y)$ viewed as a chain complex of modules ${}_\alpha C_n(\tilde{Y})$ over $Z\Xi$ by means of the action $m \cdot x = \alpha(m) \cdot x$ for $m \in Z\Xi$ and $x \in C_n(\tilde{Y})$. Any map $f: X \to Y$ with $f_\# = \alpha$ on the fundamental groups, lifts to give a map $\tilde{f}: \tilde{X} \to \tilde{Y}$ which induces a chain map $\tilde{f}_*: C_*(\tilde{X}) \to {}_\alpha C_*(\tilde{Y})$ of $Z\Xi$-module homomorphism. Conversely, any chain map $v: C_*(\tilde{X}) \to {}_\alpha C_*(\tilde{Y})$ with $v_0 = Z_\alpha: C_0(\tilde{X}) = Z\Xi \to {}_\alpha Z\pi = {}_\alpha C_0(\tilde{Y})$, is realizable by a map $f: X \to Y$ such that $f_\#: \pi_1 X \to \pi_1 Y$ is $\alpha: \Xi \to \pi$ and $Z\Xi$-module homomorphism $f_\#: \pi_2(X) \to \pi_2(Y)$ coincides with $v_2 | \ker \partial_2(\tilde{X}): \ker \partial_2(\tilde{X}) \to \ker \partial_2(\tilde{Y})$ under the identifications $\ker \partial_2(\tilde{X}) \equiv \pi_2(X)$ and $\ker \partial_2(\tilde{Y}) \equiv \pi_2(Y)$. Thus $X$ and $Y$ have the same homotopy type if and only if the above homomorphism $\alpha: \Xi \to \pi$ is an isomorphism and there is a chain map $v: C_*(\tilde{X}) \to {}_\alpha C_*(\tilde{Y})$ which restricts to $\ker \partial_2(\tilde{X})$ to give an $Z\Xi$-module isomorphism (see Schellenberg [12]).

Since $\tilde{X}$ is simply connected, the chain complex $C_*(\tilde{X})$ provides us with the truncated free resolution $\varepsilon: C_*(\tilde{X}) \to Z$ which we can extend into a free resolution

$$C_*(\Xi): \cdots \longrightarrow C_3(\Xi) \xrightarrow{\partial_3(\Xi)} C_2(\tilde{X}) \xrightarrow{\partial_2(\tilde{X})} C_1(\tilde{X}) \xrightarrow{\partial_1(\tilde{X})} C_0(\tilde{X}) \xrightarrow{\varepsilon} Z \longrightarrow 0$$

$$C_0(\tilde{X}) = Z\Xi$$

of the trivial module $Z$ over $Z\Xi$ ($\varepsilon: Z\Xi \to Z$ is the augmentation homomorphism). In view of the exactness of the resolution $C_*(\Xi)$, we have that Image $\partial_3(\Xi) = \ker \partial_2(\tilde{X}) \equiv \pi_2(X)$. Since any free resolution of the trivial module $Z$ over $Z\Xi$ is known to be uniquely determined upto chain equivalence, the cohomology depends on the fundamental group $\Xi$ alone.

The following "comparison theorem" will be helpful in the proof of Theorem 1. We state it in a more general setting than required for Theorem 1.

Let $\Xi$ and $\pi$ be two groups such that $H^3(\Xi, Z\Xi) = 0$ and $H^3(\pi, Z\pi) = 0$. Let $C_*(\Xi)$ and $C_*(\pi)$ be free resolutions of finite type (i.e., each module is finitely generated) over $Z\Xi$ and $Z\pi$, respectively, of the trivial module $Z$.

THEOREM 3. *Let $\alpha\colon \Xi \to \pi$ be a group isomorphism. If $u\colon C_*(\Xi) \to {}_\alpha C_*(\pi)$ is any chain map over $Z\Xi$ extending $1\colon Z \to Z$ and $u\colon N(\Xi) \to {}_\alpha N(\pi)$ is its restriction to kernels of $\partial_2(\Xi)$ and $\partial_2(\pi)$, the induced homomorphism*

$$u_*\colon H^3(\Xi, N(\Xi)) \longrightarrow H^3(\Xi, {}_\alpha N(\pi))$$

*is an isomorphism. Moreover, if $v$ is any other such chain map, then $u_* = v_*\colon H^3(\Xi, N(\Xi)) \to H^3(\Xi, {}_\alpha N(\pi))$.*

*Proof.* Since $C_*(\pi)$ is free over $Z\pi$, there exists a chain map $u'\colon C_*(\pi) \to {}_{\alpha^{-1}}C_*(\Xi)$ over $Z\pi$ extending $1\colon Z \to Z$, or equivalently, a chain map $u'\colon {}_\alpha C_*(\pi) \to C_*(\Xi)$ over $Z\Xi$ extending $1\colon Z \to Z$. We again denote by $u'\colon {}_\alpha N(\pi) \to N(\Xi)$ the restriction of $u_2$ to kernels of $\partial_2(\pi)$ and $\partial_2(\Xi)$. We prove that $u'_*u_* = 1_{H^3(\Xi, N(\Xi))}$. Because both $u'u$ and $1\colon C_*(\Xi) \to C_*(\Xi)$ extend the identity map, they are chain homotopic so that there exists a chain homotopy $s\colon 1 \simeq u'u$, i.e., $1 - u'u = \partial(\Xi)s + s\partial(\Xi)$. For $\{f\} \in H^3(\Xi, N(\Xi))$, we have $u'uf = f - \partial_3(\Xi)s_2f - s_1\partial_2(\Xi)f = f - \partial_3(\Xi)s_2f$ since $f\colon C_3(\Xi) \to N(\Xi) = \ker \partial_2(\Xi)$, and we have $\partial_3(\Xi)s_2f \in B^3(\Xi, N(\Xi))$ since $\{s_2f\} \in H^3(\Xi, C_2(\Xi)) = 0$, by the hypothesis $H^3(\Xi, Z\Xi) = 0$ and the fact that the functor $H^3(\Xi, -)$ is additive (i.e., it commutes with finite direct sums). Using the hypothesis $H^3(\pi, Z\pi) = 0$, one can similarly show $u_*u'_* = 1_{H^3(\Xi, {}_\alpha N(\pi))}$.

Finally let $v\colon C_*(\Xi) \to {}_\alpha C_*(\pi)$ be any other chain map over $Z\Xi$ extending $1\colon Z \to Z$. We prove that $(u - v)_*$ is the zero homomorphism. Because both $u, v\colon C_*(\Xi) \to {}_\alpha C_*(\pi)$ extend the identity map $1\colon Z \to Z$, there exists a chain homotopy $s\colon u \simeq v$, i.e., $u - v = \partial(\pi)s + s\partial(\Xi)$. For $\{f\} \in H^3(\Xi, N(\Xi))$, we have

$$\begin{aligned}(u - v)f &= \partial_3(\pi)s_2f + s_1\partial_2(\Xi)f \\ &= \partial_3(\pi)s_2f\ ,\end{aligned}$$

since $f\colon C_3(\Xi) \to N(\Xi) = \ker \partial_2(\Xi)$, and we have $\partial_3(\pi)s_2f \in B^3(\Xi, N(\pi))$ since $H^3(\Xi, C_2(\Xi)) = 0$.

In view of Lyndon's resolution, the hypothesis of the above theorem is satisfied for one-relator groups. Indeed there is a rather large class of groups for which the hypothesis holds (see [3]).

Before we can give a proof of Theorem 1, we need one more observation.

Each finite presentation

$$\mathscr{P} = (g_1, \cdots, g_m: \gamma_1, \cdots, \gamma_n)$$

of $\pi$ has a *cellular model* $C(\mathscr{P})$ with fundamental group $\pi_1(C(\mathscr{P})) = \pi$. This model is obtained from a sum $VS^1_i$ 1-spheres $S^1$, one for each generator $g_i$, by attaching 2-cells via maps $S^1 \to VS^1_i$ spelling out the relators $\gamma_j$. Using the standard argument for collapsing a maximal tree, each finite connected 2-dimensional $CW$-complex has the homotopy tpye of the cellular model $C(\mathscr{P})$ of some finite presentation $\mathscr{P}$ of $\pi = \pi_1 X$.

*Proof of Theorem* 1. Let $X$ and $Y$ be finite connected 2-dimensional $CW$-complexes with a single 2-cell and isomorphic fundamental groups. Since $X$ and $Y$ have the same homotopy type as the cellular models $C(\mathscr{R})$ and $C(\mathscr{Q})$, respectively, where

$$\mathscr{R} = (x_1, \cdots, x_n: R^r)$$

and

$$\mathscr{Q} = (y_1, \cdots, y_m: Q^q)$$

($R$ and $Q$ are not proper powers) are finite presentations for $\Xi = \pi_1 X$ and $\pi = \pi_1 Y$, we may assume that $X = C(\mathscr{R})$ and $Y = C(\mathscr{Q})$.

Suppose $r = 1$. Then $\Xi$ is torsion-free ([11], Theorem 4.2, p. 266). This implies that $\pi$ is torsion-free as well so that $q = 1$; thus $X$ and $Y$ are aspherical (see [10], [1], or [4]). Since by hypothesis $\pi_1(X) = \Xi \approx \pi = \pi_1(Y)$, they have the same homotopy type and in fact there is a homotopy equivalence between $X$ and $Y$ inducing any isomorphism $\alpha: \Xi \to \pi$.

Thus we assume $r \geqq 2$. We claim that $r = q$ and $n = m$. The first follows since $R$ defines an element exactly of order $r$ in $\Xi$ ([11], Corollary 4.11, p. 266) and elements of finite order in $\Xi$ and $\pi$ are defined by conjugates of powers of $R$ and $Q$, respectively, ([11], Theorem 4.13, p. 269). The second follows by looking at the abelianizations of the two groups.

Now let $\alpha: \Xi \to \pi$ be any given isomorphism. Then $\alpha(R) = gQ^t g^{-1}$ where $g \in \pi$, $(t, r) = 1$ ([11], Theorem 4.13, p. 269). Because $X = C(\mathscr{R})$ and $Y = C(\mathscr{Q})$, the truncated free resolutions $\varepsilon: C_*(\tilde{X}) \to Z$ and $\varepsilon': C_*(\tilde{Y}) \to Z$ coincide with the initial segments of Lyndon's resolutions $C_*(\Xi)$ and $C_*(\pi)$ of the trivial module $Z$ over $Z\Xi$ and $Z\pi$, respectively (see §1). Thus we obtain

$$\begin{array}{l} C_*(\Xi)\colon \cdots \xrightarrow[\langle R,r\rangle]{\partial_4(\Xi)} C_3(\Xi) \xrightarrow[R-1]{\partial_3(\Xi)} C_2(\tilde{X}) \xrightarrow{\partial_2(\tilde{X})} C_1(\tilde{X}) \xrightarrow{\partial_1(\tilde{X})} C_0(\tilde{X}) \xrightarrow{\varepsilon} Z \longrightarrow 0 \\ \qquad\qquad\qquad\quad \| \qquad\qquad\quad \| \qquad\qquad\quad \| \qquad\qquad \| \\ \qquad\qquad\qquad\; Z\Xi \qquad\quad\; Z\Xi \qquad\; (Z\Xi)^n \qquad\; Z\Xi \end{array}$$

and

$$\begin{array}{l} C_*(\pi)\colon \cdots \xrightarrow[\langle Q,r\rangle]{\partial_4(\pi)} C_3(\pi) \xrightarrow[Q-1]{\partial_3(\pi)} C_2(\tilde{Y}) \xrightarrow{\partial_2(\tilde{Y})} C_1(\tilde{Y}) \xrightarrow{\partial_1(\tilde{Y})} C_0(\tilde{Y}) \xrightarrow{\varepsilon'} Z \longrightarrow 0\,. \\ \qquad\qquad\qquad\quad \| \qquad\qquad\quad \| \qquad\qquad\quad \| \qquad\qquad \| \\ \qquad\qquad\qquad\; Z\pi \qquad\quad\; Z\pi \qquad\; (Z\pi)^n \qquad\; Z\pi \end{array}$$

As usual we invoke identifications $\pi_2(X) \equiv Z\Xi(R-1)$ and $\pi_2(Y) \equiv Z\pi(Q-1)$.

Let $u\colon C_*(\Xi) \to {}_\alpha C_*(\pi)$ be any chain map extending the identity map $1\colon Z \to Z$ and let $u$ also denote the restriction $u_2 | Z\Xi(R-1)\colon Z\Xi(R-1) \to {}_\alpha Z\pi(Q-1)$. From Theorem 3, we have that $u_*\colon H^3(\Xi, Z\Xi(R-1)) \to H^3(\Xi, {}_\alpha Z\pi(Q-1))$ is an isomorphism.

Then $Z\Xi$-module isomorphism

$$Z\Xi \xrightarrow{Z\alpha} {}_\alpha Z\pi \xrightarrow{g} {}_\alpha Z\pi$$

carries $(R-1)$ to $g(Q^t-1)$ and hence induces a $Z\Xi$-module isomorphism

$$w\colon Z\Xi(R-1) \longrightarrow {}_\alpha Z\pi(Q-1)$$

since $Z\pi g(Q^t-1) = Z\pi(Q^t-1) = Z\pi(Q-1)$ [by Lemma 2 (i)]. Because $w_*\colon H^3(\Xi, Z\Xi(R-1)) \to H^3(\Xi, {}_\alpha Z\pi(Q-1))$ is an isomorphism, we obtain an isomorphism $\bar{w}\colon H^3(\Xi, Z\Xi(R-1)) \to H^3(\Xi, Z\Xi(R-1))$ such that $w_*\bar{w} = u_*$. Since $H^3(\Xi, Z\Xi(R-1)) \approx Z_r$ [by Lemma i], $\bar{w}$ is completely determined by its image $\bar{w}(1) = s \bmod r$ where $(s, r) = 1$. Then by Lemmas 1 and 2, $\bar{w}$ coincides with the cohomology isomorphism induced by the $Z\Xi$-module isomorphism $\langle R, s\rangle\colon Z\Xi(R-1) \to Z\Xi(R-1)$. Hence $v = w\langle R, s\rangle$ is an isomorphism from $Z\Xi(R-1) \to {}_\alpha Z\pi(Q-1)$ such that $v_* = u_*$. This means that there exists a module homomorphism $\gamma\colon C_2(\tilde{X}) = Z\Xi \to {}_\alpha Z\pi(Q-1) = \ker \partial_2(\tilde{Y})$ such that $(v-u)\circ\partial_3(\Xi) = \gamma\circ\partial_3(\Xi)$. Then $u_2 + \gamma\colon C_2(\tilde{X}) = Z\Xi \to {}_\alpha Z\pi = C_2(\tilde{Y})$ restricts to the second homotopy module $Z\Xi(R-1)$ to give $v\colon Z\Xi(R-1) \to {}_\alpha Z\pi(Q-1)$ since $(u_2+\gamma)\circ\partial_3(\Xi) = u_2\circ\partial_3(\Xi) + v\circ\partial_3(\Xi) - u\circ\partial_3(\Xi) = v\circ\partial_3(\Xi)$.

The homomorphisms $u_0 = Z\alpha$, $u_1$, and $u_2 + \gamma$ constitute a chain map $C_*(\tilde{X}) \to {}_\alpha C_*(\tilde{Y})$ which induces an isomorphism on $\ker \partial_2(\tilde{X})$.

Therefore by the preliminary remarks in this section there exists a map $f: X \to Y$ which realizes this new chain map and any such realization is actually a homotopy equivalence. This completes the proof of Theorem 1.

3. **Factorization as sums.** Let $X$ be a finite connected 2-dimensional $CW$-complex with a single 2-cell. In this section we consider homotopy factorizations of $X$ into finite sums. Since any summand in such a factorization is dominated by the connected $CW$-complex $X$, the summand has the homotopy type of a connected $CW$-complex. Hence we may always assume each summand to be a $CW$-complex. Moreover we may assume $X$ is the cellular model $C(\mathscr{P})$ of a finite presentation

$$\mathscr{P} = (x_1, \cdots, x_n: Q^q)$$

(where $Q$ is not a proper power) for $\pi = \pi_1 X$.

LEMMA 3. (i) $X \not\simeq W \vee S^2$.

(ii) *If* $X \simeq W \vee Z$ *where* $W$ *and* $Z$ *are not contractible, then* $\pi_1 W \neq 1$ *and* $\pi_1 Z \neq 1$.

*Proof.* (i) Let $f: X \to W \vee S^2$ be a homotopy equivalence. If $q = 1$, then $X = C(\mathscr{P})$ is aspherical so that $0 = \pi_2 X \approx \pi_2(W \vee S^2) \approx \pi_2 W \oplus Z\pi$, which is a contradiction. Thus we assume $q > 1$. In this case we have $Z\pi(Q-1) \approx \pi_2 X \approx \pi_2(W \vee S^2) \approx \pi_2(W) \oplus Z\pi$. But this is impossible since we have the following commutive diagram:

$$\begin{array}{ccccc} Z\pi(Q-1) \equiv \pi_2 X & \xrightarrow{f_\#} & \pi_2(W \vee S^2) \equiv \pi_2 W \oplus Z\pi \\ \downarrow h & & \downarrow \bar{h} \\ H_2 X & \xrightarrow{f_*} & H_2(W \vee S^2) \end{array}$$

where $h$ and $\bar{h}$ denote the Hurewicz homomorphisms. Here $h$ and $\bar{h}$ are given by the augmentation homomorphism $\varepsilon: Z\pi \to Z$. Clearly then $h$ is the zero homomorphism whereas $\bar{h}$ is a nonzero homomorphism, yielding a contradiction.

(ii) Suppose (ii) is not true, then without loss of generality we may assume that $\pi_1 Z = 1$. Since $X$ is 2-dimensional, $H_i X = 0$ for $i \geqq 3$ which implies that $H_i Z = 0$ for $i \geqq 3$. Furthermore since $H_2 X$ is a free abelian group of rank 0 or 1, we conclude that $H_2 Z = 0$ or $Z$. If $H_2 Z = 0$, we have that $Z$ is contractible, a contradiction. Thus assume $H_2 Z = Z$. But then $Z$ is a Moore space $M(Z, 2)$, hence $Z \simeq S^2$. This gives $X \simeq W \vee S^2$, contrary to part (i) above.

*Proof of Theorem* 2. Let us assume that $X \simeq W \vee Z$ where $W$

and $Z$ are noncontractible. Because $X = C(\mathscr{P})$ where $\mathscr{P} = (x_1, \cdots, x_n: Q^q)$, $\pi = F/R$ where $F = F(x_i)$ is the free group generated by $x_1, \cdots, x_n$ and $R$ is the normal closure of the single relator $Q^q$. Since $\pi_1 X \approx \pi_1 W * \pi_1 Z$ with $\pi_1 W \neq 1$, $\pi_1 Z \neq 1$ [by Lemma 3 (ii)], we have an epimorphism $\bar{\varphi}: F \to \pi_1 W * \pi_1 Z$ given by

$$F \xrightarrow{\theta} F/R \xrightarrow{\varphi} \pi_1 W * \pi_1 Z$$

where $\theta: F \to F/R$ is the canonical homomorphism and $\varphi: F/R = \pi_1 X \to \pi_1 W * \pi_1 Z$ is an isomorphism. Therefore by Grushko's theorem (see Kurosh [9]), there exists generators $w_1, \cdots, w_l, z_1, \cdots, z_k$ of $F$ such that $\bar{w}_i = \bar{\varphi}(w_i)$ generate $\pi_1 W$ and $\bar{z}_j = \bar{\varphi}(z_j)$ generate $\pi_1 Z$. Thus $\pi$ has presentation

$$(w_1, \cdots, w_l, z_1, \cdots, z_k: r(w_i, z_j))$$

where $r(w_i, z_j)$ is the original relator $Q^q \in F(x_i) = F(w_i, Z_j)$ written in terms of the now generators.

We claim that $r(w_i, z_j)$ is a reduced word either in $w_i$ or in $z_j$ only. To see this suppose $r = r(w_i, z_j)$ involves both $w_i$'s and $z_j$'s. We can write $r \neq 1$ in $F(w_i, z_j)$ uniquely as a product $V_1, \cdots, V_s$ where $V_t \in F(w_i)$ or $F(z_j)$, $V_t \neq 1$ and such that $V_t$ and $V_{t+1}$ belong to different factors of the free product $F(w_i) * F(z_j)$. Since $\bar{\varphi}(r) = 1$ in $\pi_1 W * \pi_1 Z$, it follows that for some index $v$, $1 \leqq v \leqq s$, $\bar{\varphi}(V_v) = 1$ in $\pi_1 W$ or in $\pi_1 Z$. Without loss of generality, suppose $V_v(\bar{w}_i) = \bar{\varphi}(V_v) = 1$ in $\pi_1 W$ so that $V_v(w_i) = 1$ in $\pi$. But this is impossible: the single relator $r$ does involve $z_j$, hence by the Freiheitssatz ([11], Theorem 4.1, p. 252) the subgroup of $\pi = F/R$ generated by the generators $w_i$ is freely generated by them so that $V_v(w_i) \neq 1$ in $\pi$.

Thus we may assume that the original relator $r$ is a word in only $w_i$. Hence $\pi_1 Z$ is presented by $(z_1, \cdots, z_k:)$ and $\pi_1 W$ is presented by $(w_1, \cdots, w_l: r(w_i))$, and the original isomorphism $\varphi$ is a factor-wise isomorphism

$$\varphi = \varphi_W * \varphi_Z: F/R = F(w_i)/N(r(w_i)) * F(z_j) \longrightarrow \pi_1 W * \pi_1 Z$$

where $N(r(w_i))$ is the normal closure in $F(w_i)$ of the single relator $r(w_i)$ and $F(z_j)$ is the free group of rank $k$ generated by $z_1, \cdots, z_k$.

Therefore $\pi_1 Z$ is a free group of rank $k$ and since $Z$ is a retract of a 2-dimensional $CW$-complex $X$, by a result of C. T. C. Wall ([14], Proposition 3.3), $Z$ has the homotopy type of a finite bouquet of 1-spheres and 2-spheres. But in view of Lemma 3 (i), there can be no 2-spheres involved; therefore $Z \simeq kS^1$.

By Theorem 1, there is a homotopy equivalence

$$f: W \vee kS^1 \longrightarrow Y \vee kS^1$$

where $Y$ is the cellular model of the presentation $(w_1, \cdots, w_l: r(w_i))$ and $f_\# = \varphi_W * 1: \pi_1 W * F^k \to \pi_1 Y * F^k$. Now we can attach $k$ 2-cells via the attaching maps which are identity on the $k$ 1-spheres, and the homotopy equivalence $f$ extends to a homotopy equivalence $W \vee kB^2 \simeq Y \vee kB^2$ ([7], Prop. 6.8, p. 41). Thus $W \simeq Y$.

Finally let us assume that $\pi_1 X \approx H * K$ with $H \neq 1$ and $K \neq 1$. Without loss of generality we may assume that $H$ is a one-relator group and $K$ is a free group of rank $k$, say. Then by Theorem 1, $X \simeq W \vee Z$ where $W$ is the cellular model of a single relator presentation for $H$ and $Z = kS^1$. This completes the proof.

**4. An example.** One might attempt to generalize Theorem 1 to 2-dimensional $CW$-complexes with one-relator fundamental groups but having more than a single 2-cell. Unfortunately, we have the following example of Dunwoody [2] which involves homotopically distinct 2-dimensional $CW$-complexes with two 2-cells and isomorphic one-relator fundamental groups. Namely he has shown that the cellular models of the presentations

$$\mathscr{P} = (a, b: a^2b^{-3}, 1)$$

and

$$\mathscr{R} = (a, b: (a^2b^{-3})(a^2b^{-3})^a(a^2b^{-3})^{a^2}, (a^2b^{-3})(a^2b^{-3})^b(a^2b^{-3})^{b^2}(a^2b^{-3})^{b^3})$$

of the trefoil group do not have the same homotopy type ($x^g$ denotes $g^{-1}xg$). However $C(\mathscr{P}) \vee S^2 \simeq C(\mathscr{R}) \vee S^2$.

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